

Quantum Computing with Heegaard Splittings of 3-manifolds

Henrique ENNES

Advisors: Clément MARIA and Nicolas NISSE

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Introduction

Part 1: Topology

- Heegaard splittings
- ⇒ Data structure for Heegaard splittings
- ⇒ Restricting manifold topology

Part 2: Quantum computing

- Quantum circuits
- ⇒ Geometric Solovay-Kitaev
- ⇒ Flags for error detection

Conclusion

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- $\implies$ Data structure for Heegaard splittings
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Conclusion

Manifolds

- **Manifold:** locally $\mathbb{R}^d$



$d = 2 \rightarrow$ surface

- **Closed manifold:** compact + no boundary



$d = 3$, not-closed

Manifolds

- **Manifold:** locally $\mathbb{R}^d$

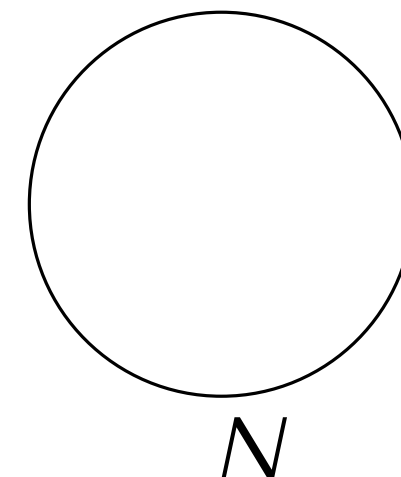
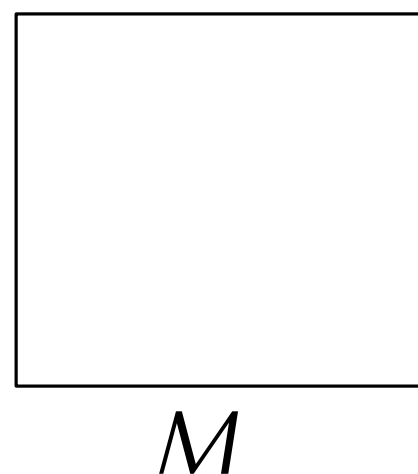


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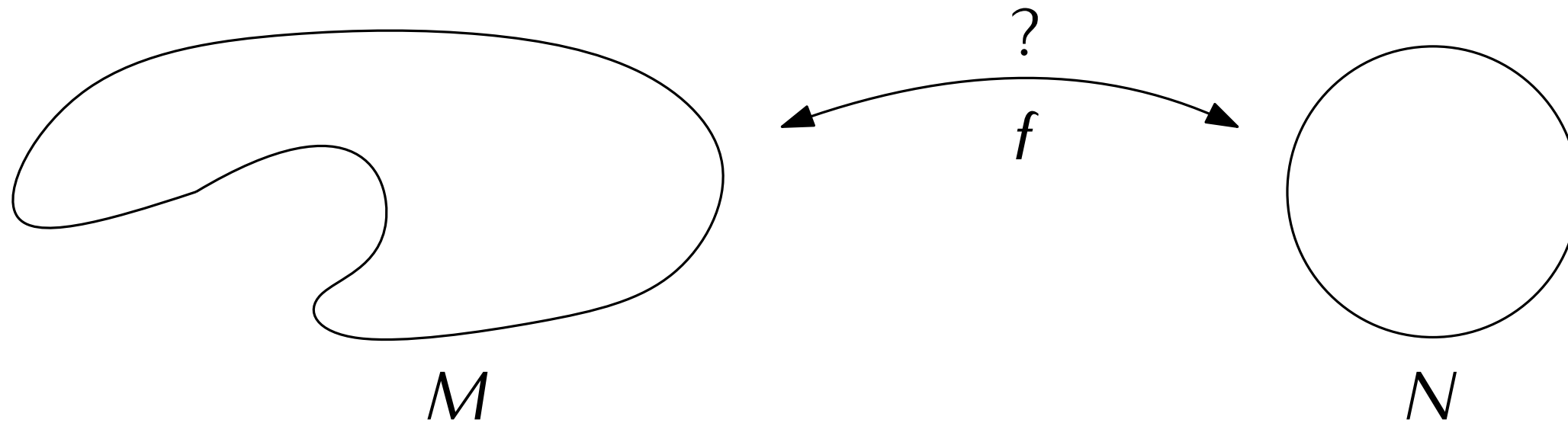
- **Closed manifold:** compact + no boundary
- **Homeomorphism:** $f : M \rightarrow N$ and f^{-1} are continuous



The homeomorphism problem

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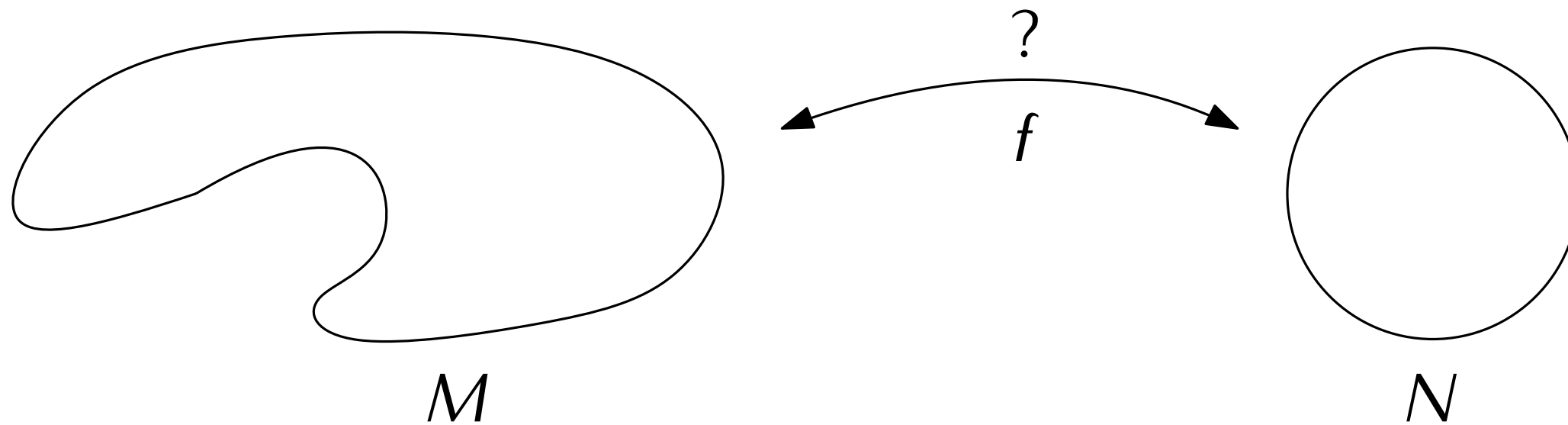
Problem: are M and N homeomorphic?



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Dimension 2

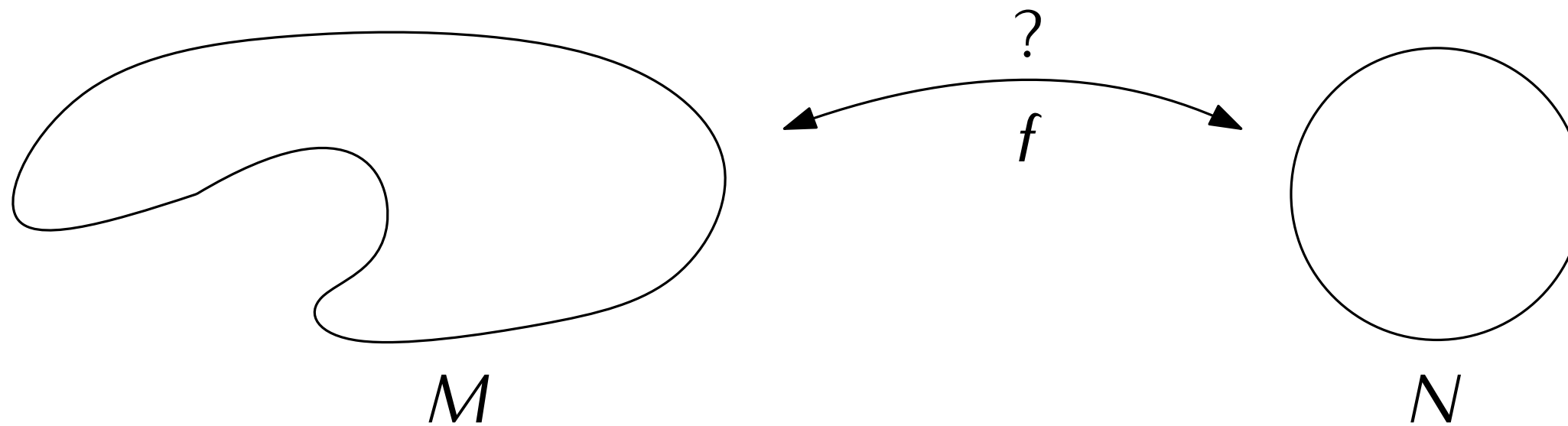
Dimension 3

Dimension ≥ 4

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Dimension 2 [classification of surfaces (Brahana, 1921)] P

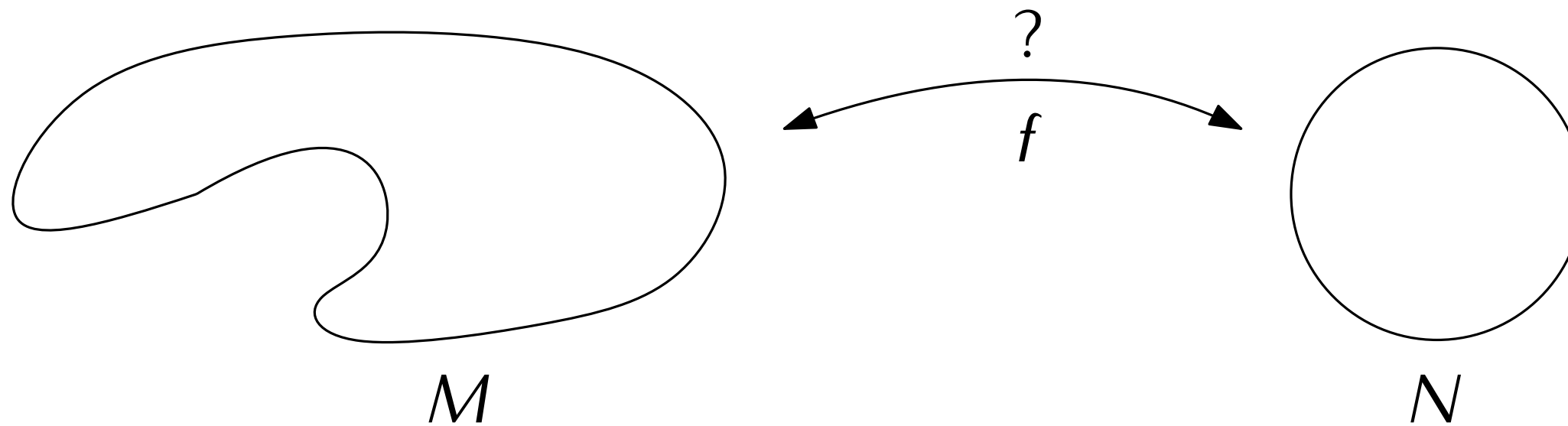
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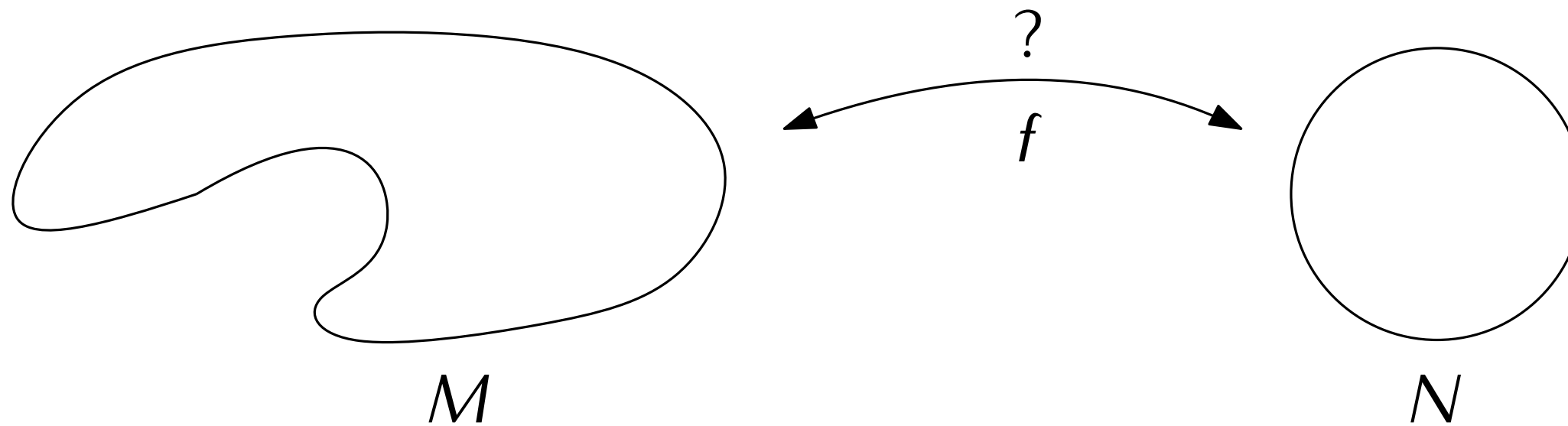
Dimension 3

Dimension ≥ 4 [Markov, 1958] UNDECIDABLE

The homeomorphism problem

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Problem: are M and N homeomorphic?



Dimension 2 [classification of surfaces (Brahana, 1921)] P

Dimension 3 [Perelman, 2003] DECIDABLE

Dimension ≥ 4 [Markov, 1958] UNDECIDABLE

Invariants

- **Invariants:** $M = N \implies \langle M \rangle = \langle N \rangle$
- **Distinguishing invariants:** for all M, N , $\langle M \rangle = \langle N \rangle$ “often” implies $M = N$
- **Reshetikhin–Turaev (RT):** distinguishing quantum invariants of **closed 3-manifolds**

Invariants

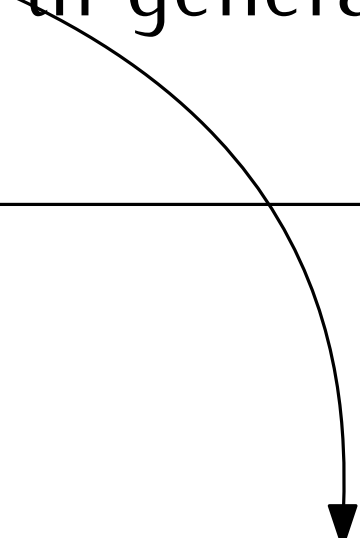
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Alagic–Lo [2014]: #P-hard in general to even approximate RT invariants of closed 3-manifolds

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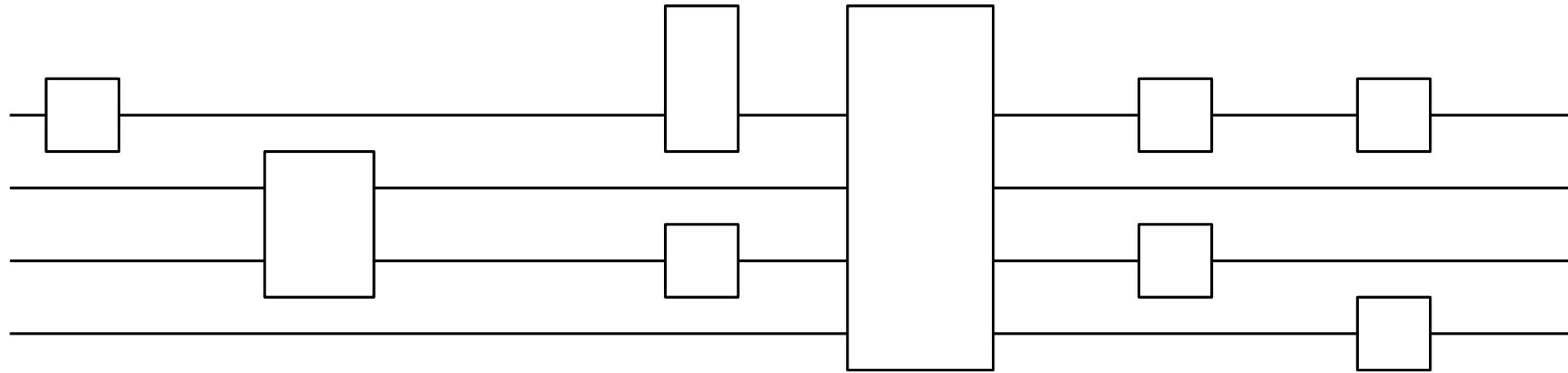
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count # solutions in SAT

Proving Alagic-Lo Theorem

Quantum circuit

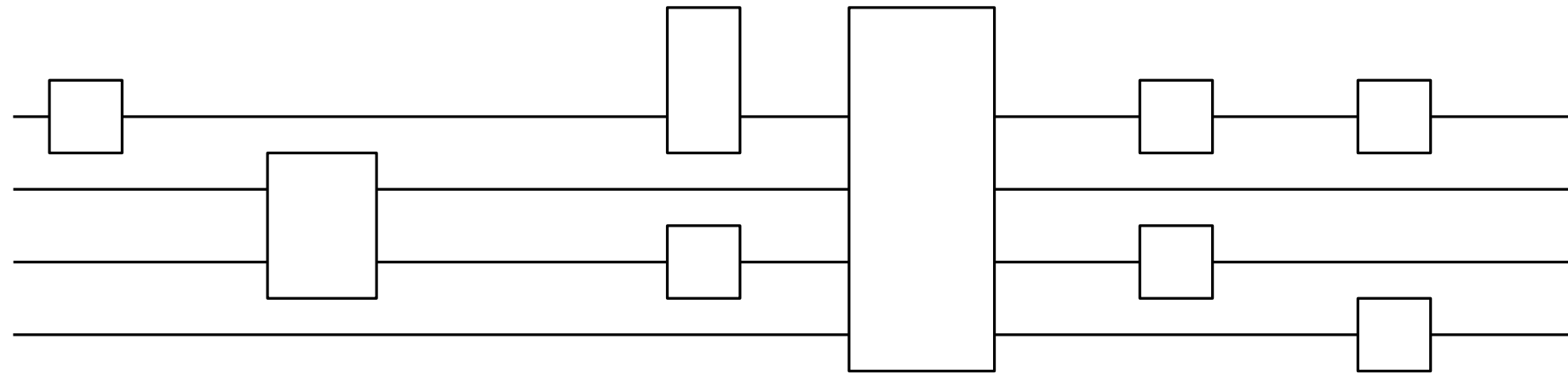


efficient
reduction

3-manifold

Proving Alagic-Lo Theorem

Quantum circuit



[Aaronson, 2005] “Evaluating”
a quantum circuit is #P-hard

$\approx$

efficient
reduction

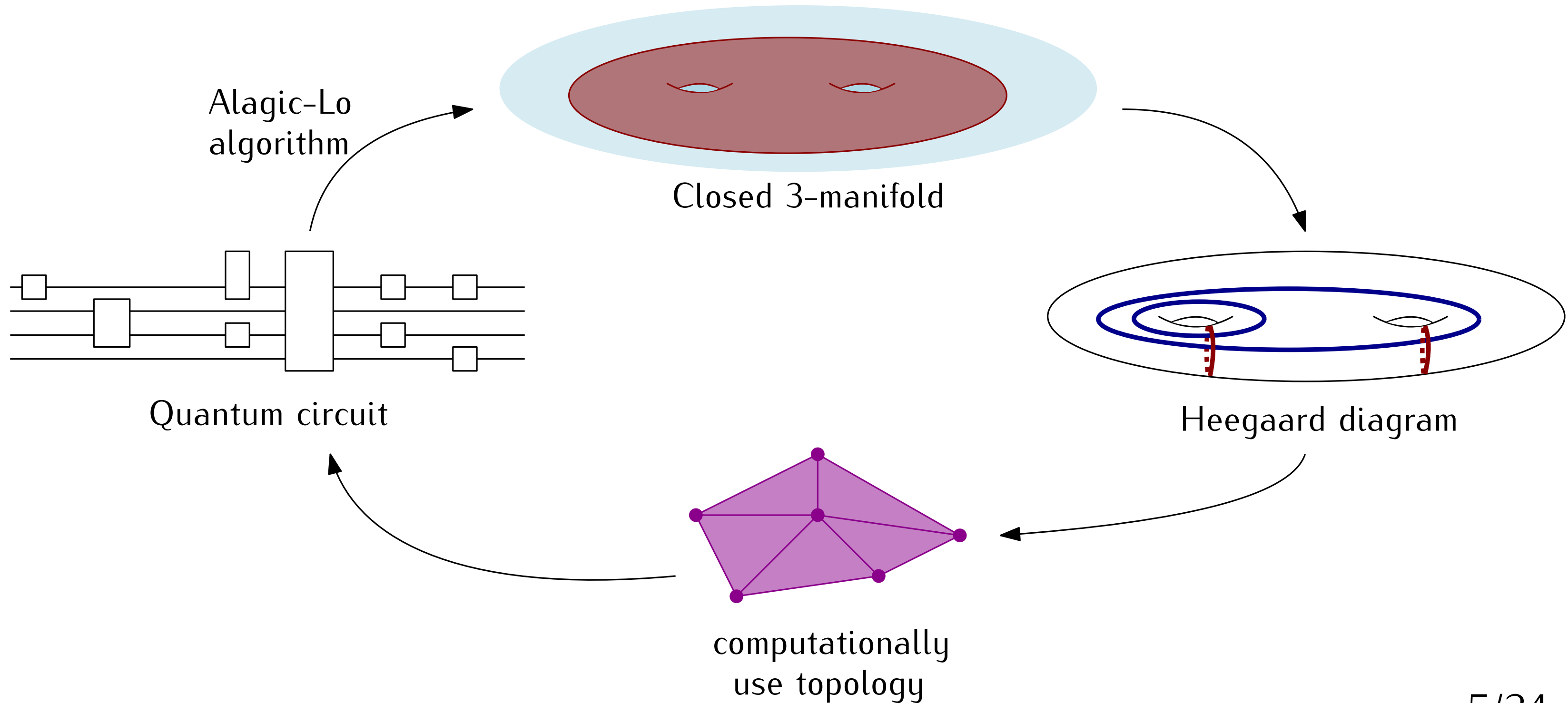


3-manifold

[Alagic & Lo, 2014] Approximating
the invariant needs also to be
#P-hard!

Our roadmap

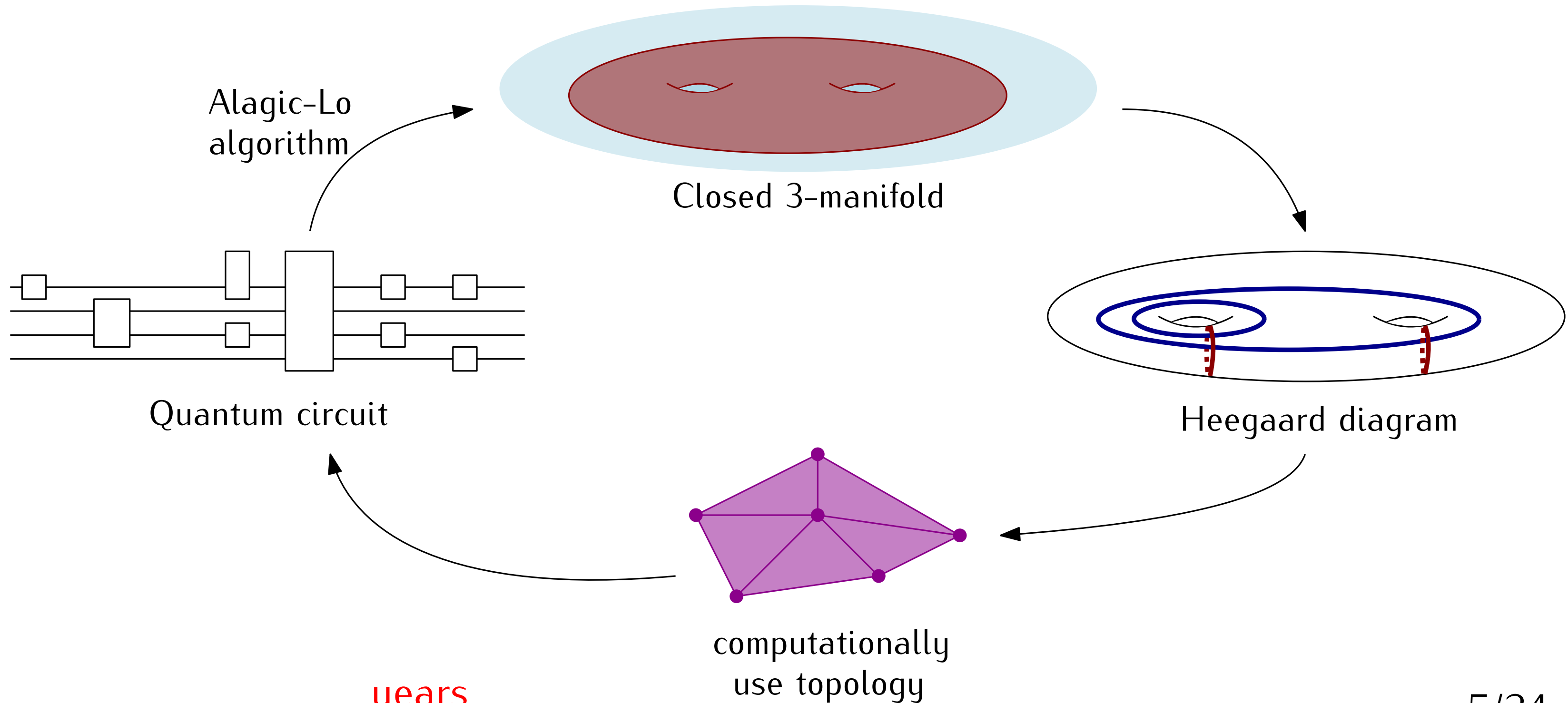
Can we **implement** the Alagic-Lo algorithm?



We can do it in 3 months!

Our roadmap

Can we **implement** the Alagic-Lo algorithm?



We can do it in 3 ~~months~~ ^{years}!

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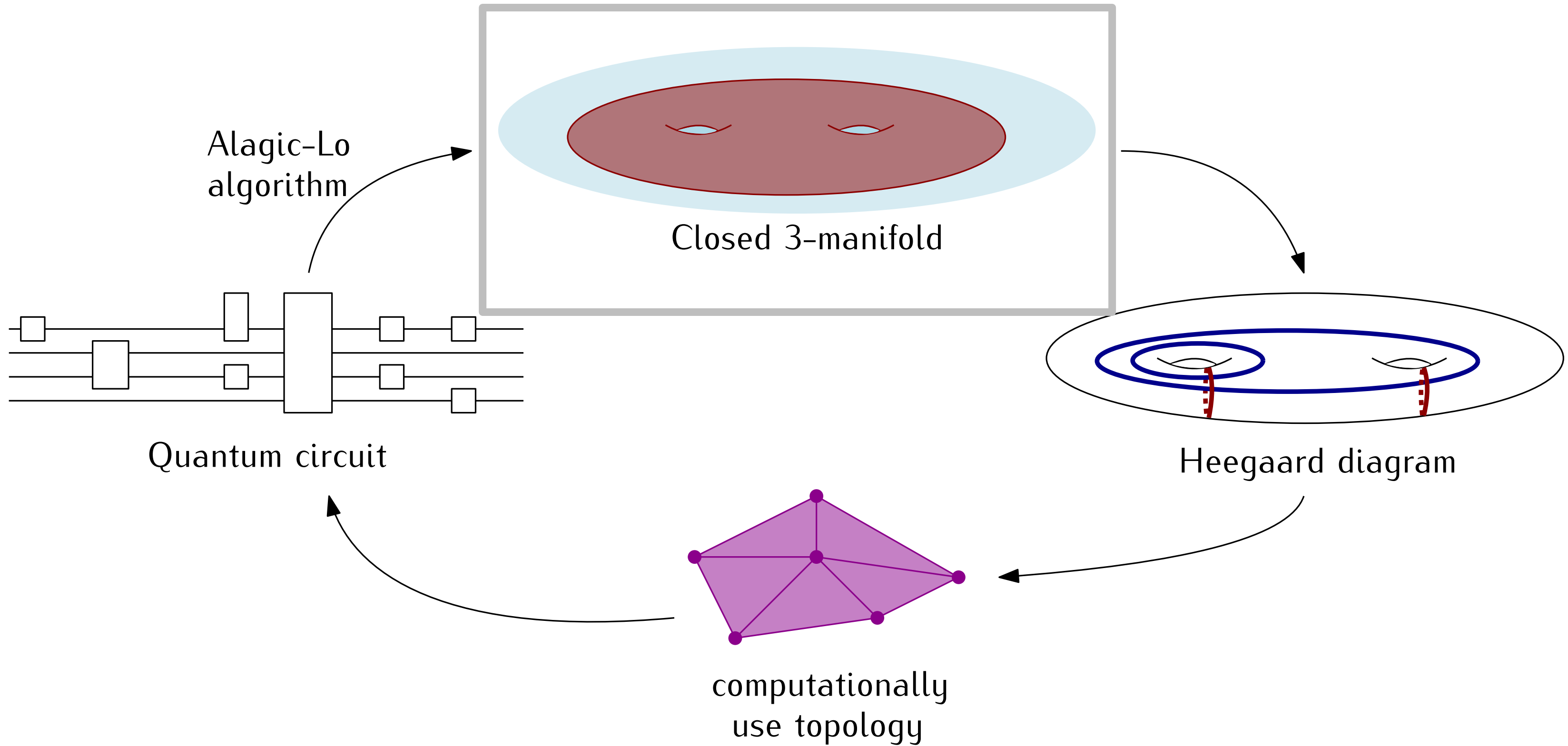
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- ⇒ Restricting manifold topology

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- ⇒ Flags for error detection

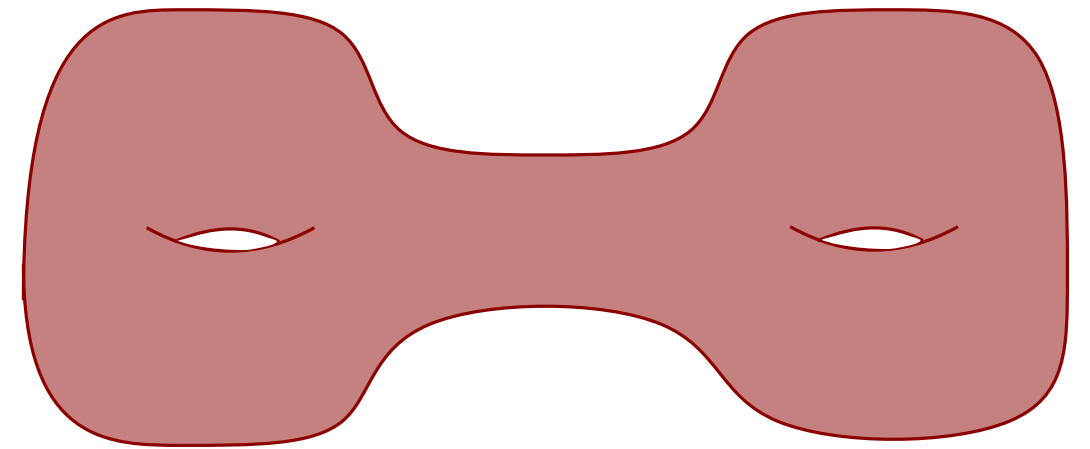
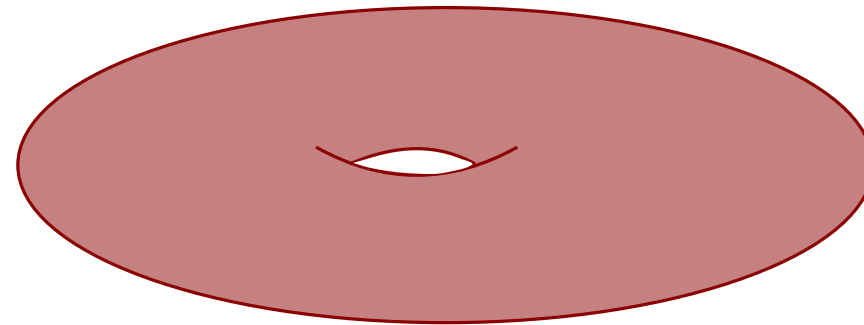
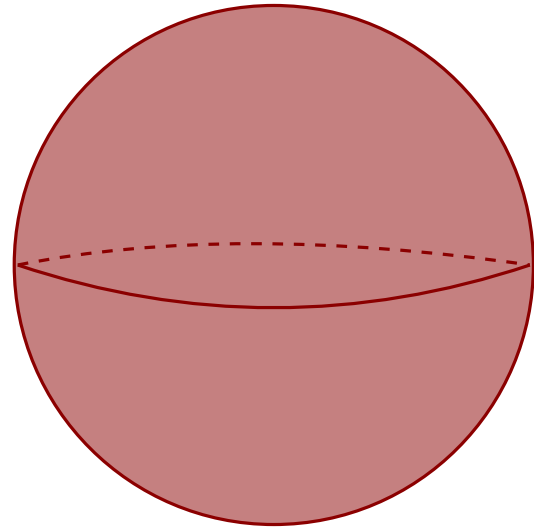
Conclusion

Roadmap: Heegaard splittings



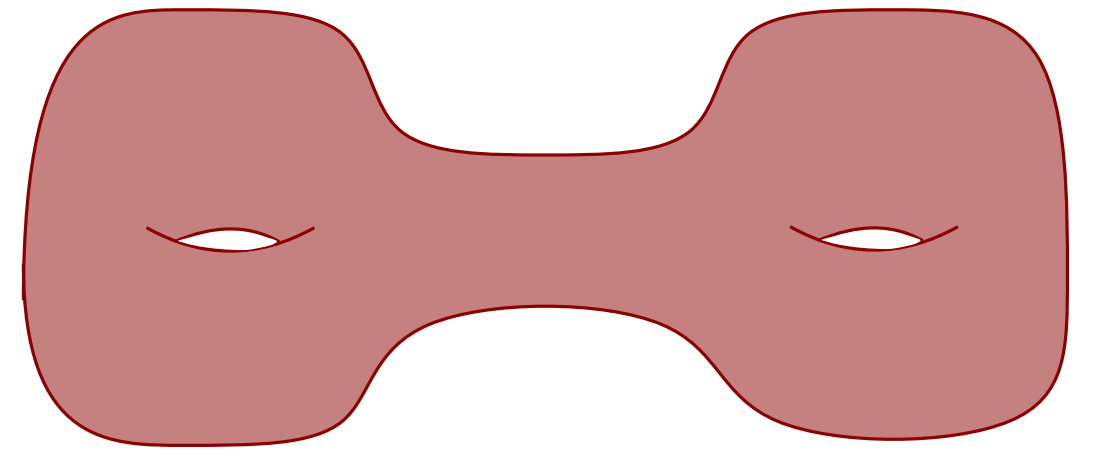
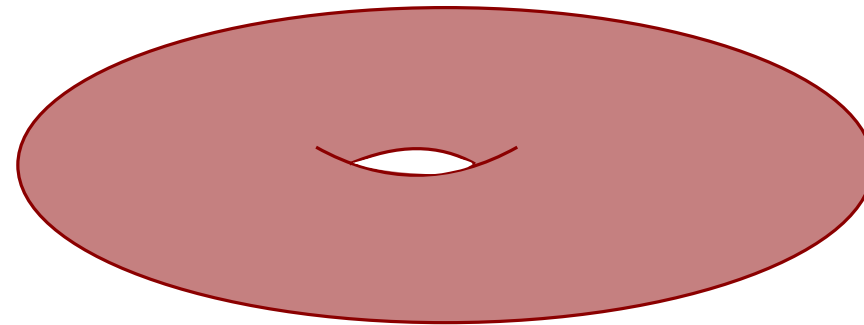
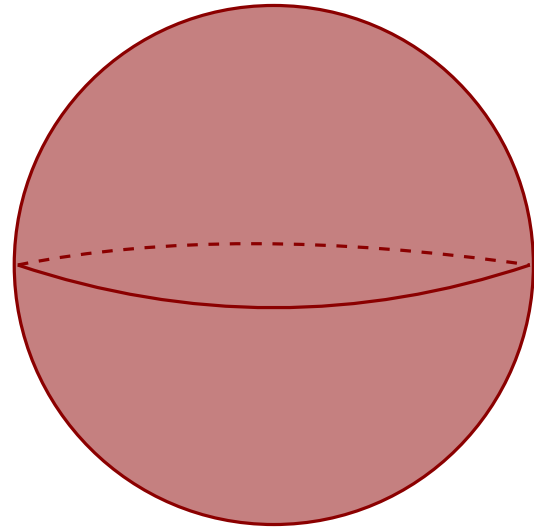
Handlebodies

- 3-handlebody: “solid” surface in $\mathbb{R}^3$

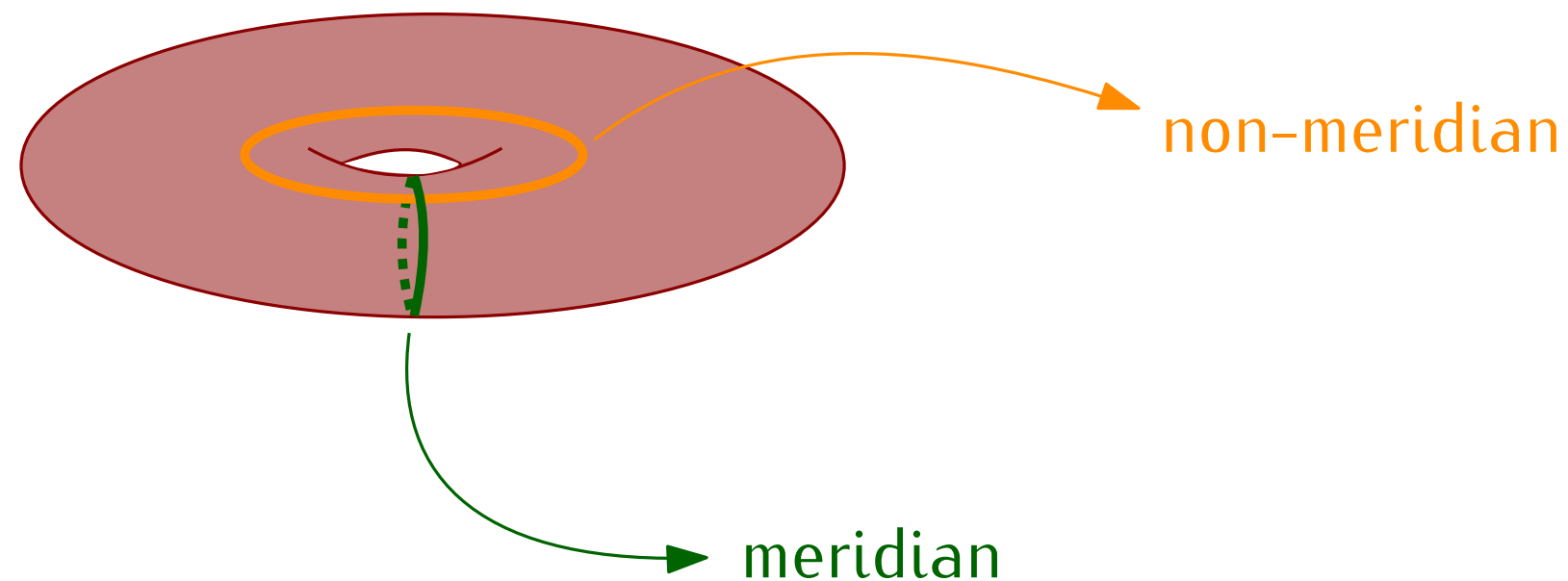


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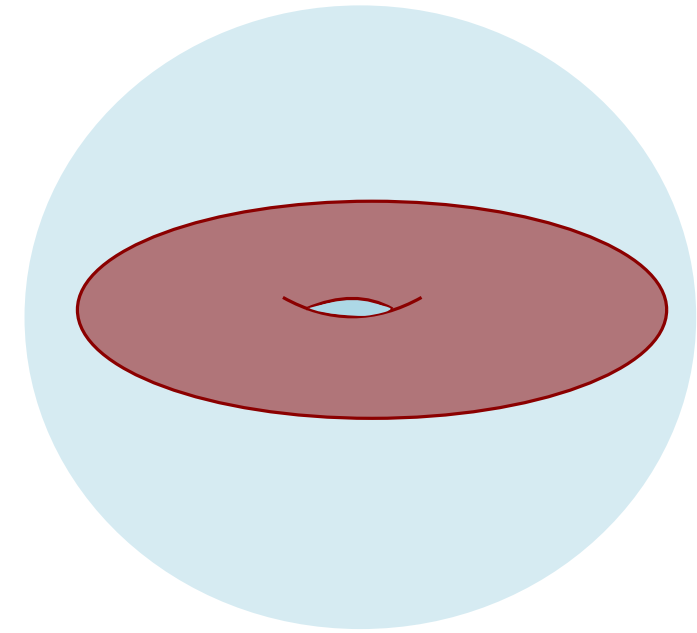


- Meridian: curve that bounds a disk inside the handlebody



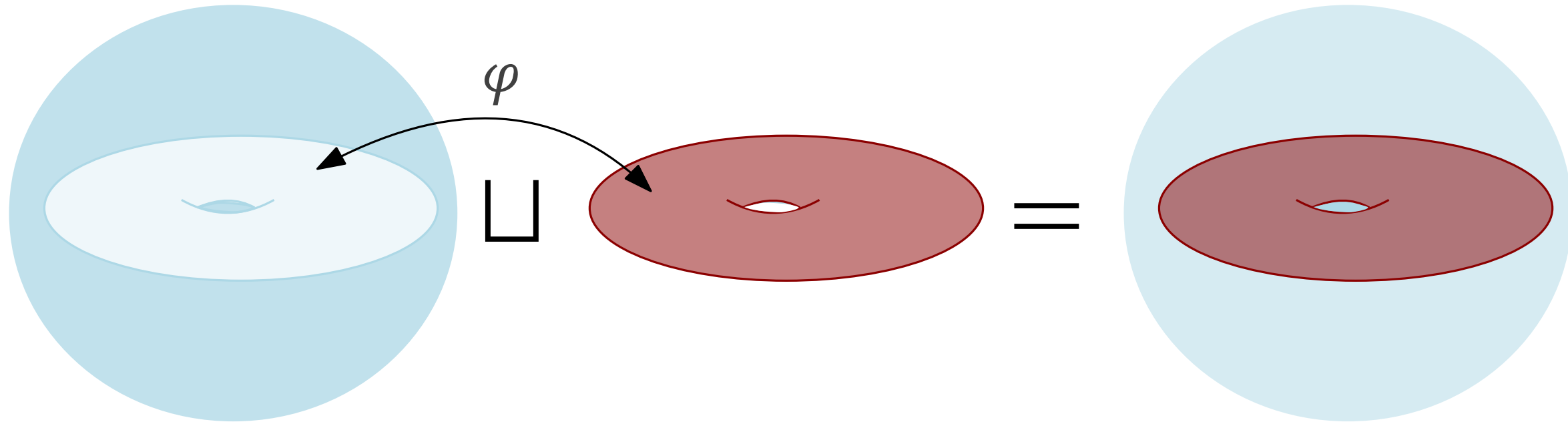
Heegaard splittings

- Sphere S^3 : glue two solid tori



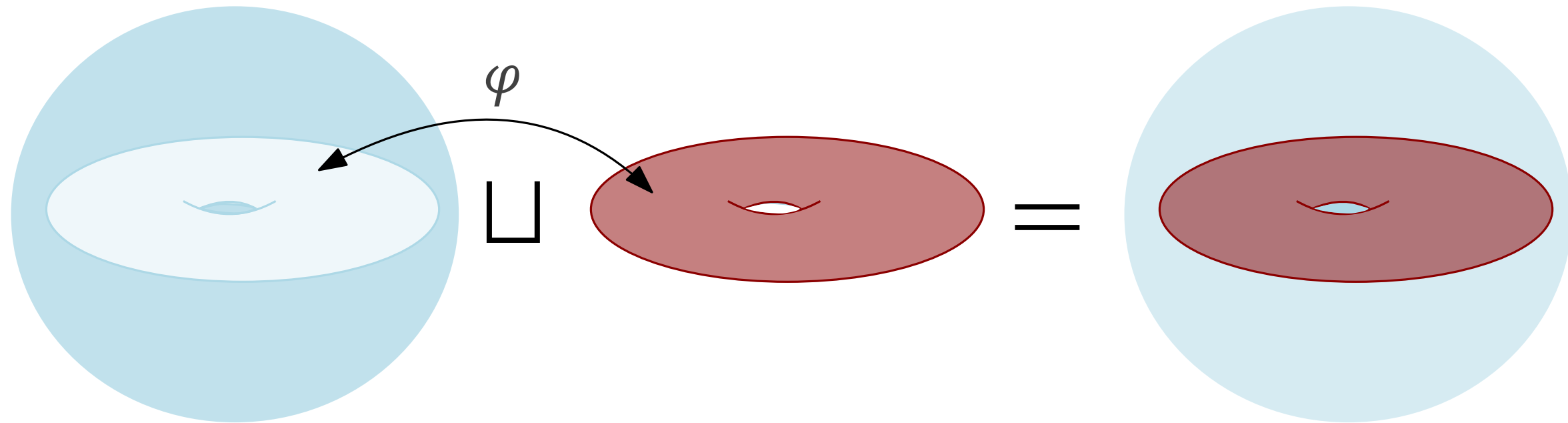
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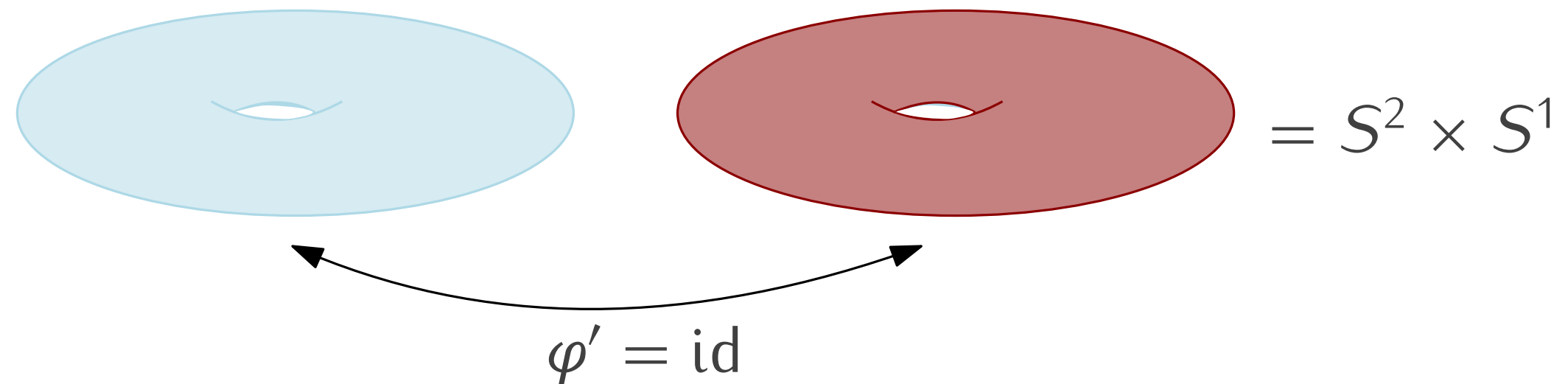


Heegaard splittings

- Sphere S^3 : glue two solid tori



- Changing the gluing map gives **different manifolds**



Heegaard splittings

Heegaard splitting: two copies of a handlebody and a gluing map
(= homeomorphism from boundary to boundary)

Every closed 3-manifold has a Heegaard splitting

Heegaard splittings

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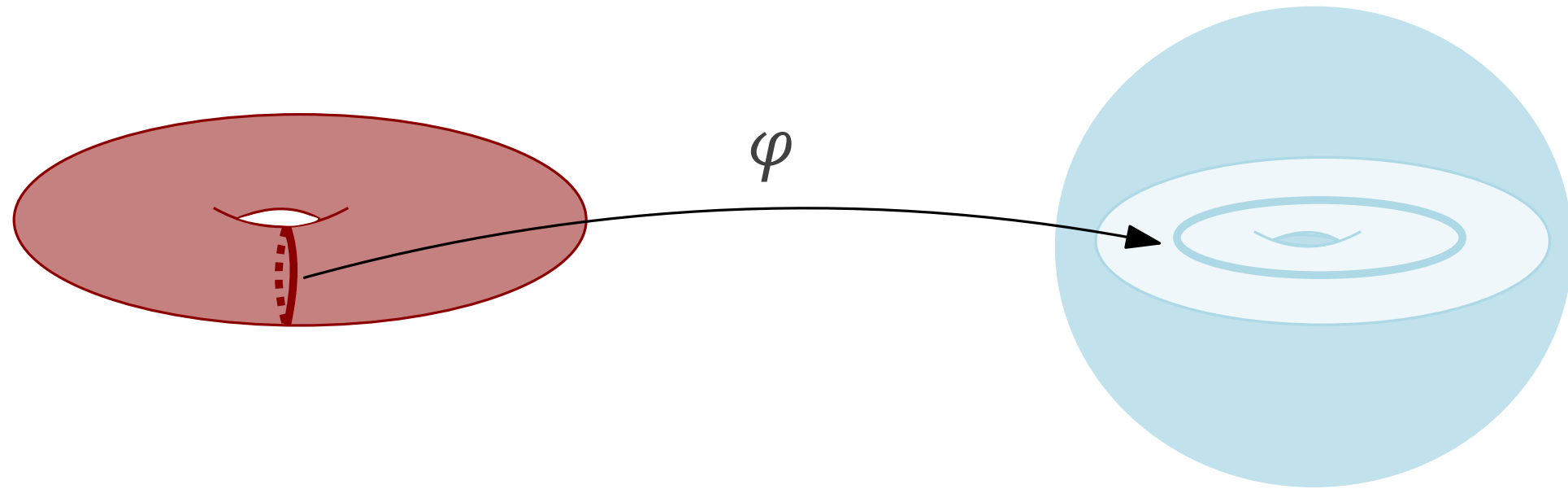
Every closed 3-manifold has a Heegaard splitting



continuous data

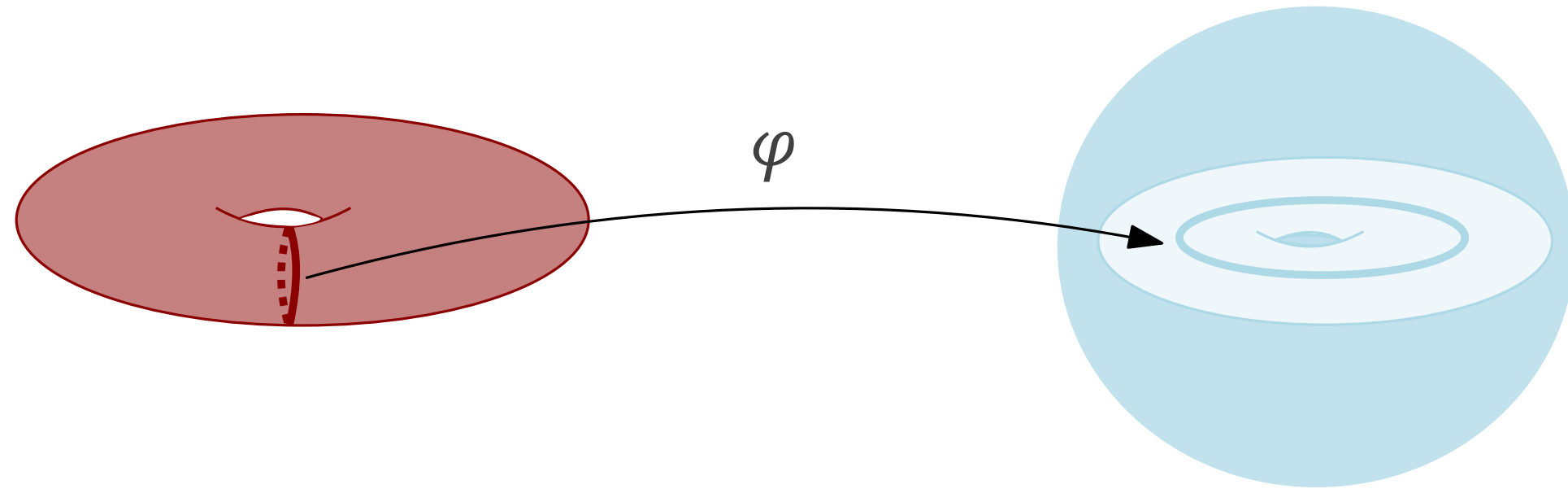
Heegaard diagrams

Gluing map takes meridians of one handlebody to meridians of the other handlebody

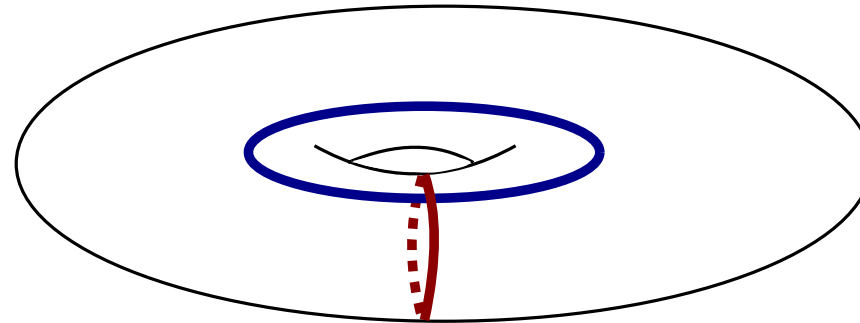


Heegaard diagrams

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- Heegaard diagram: gluing surface + two sets of meridians



Every closed 3-manifold has a Heegaard diagram

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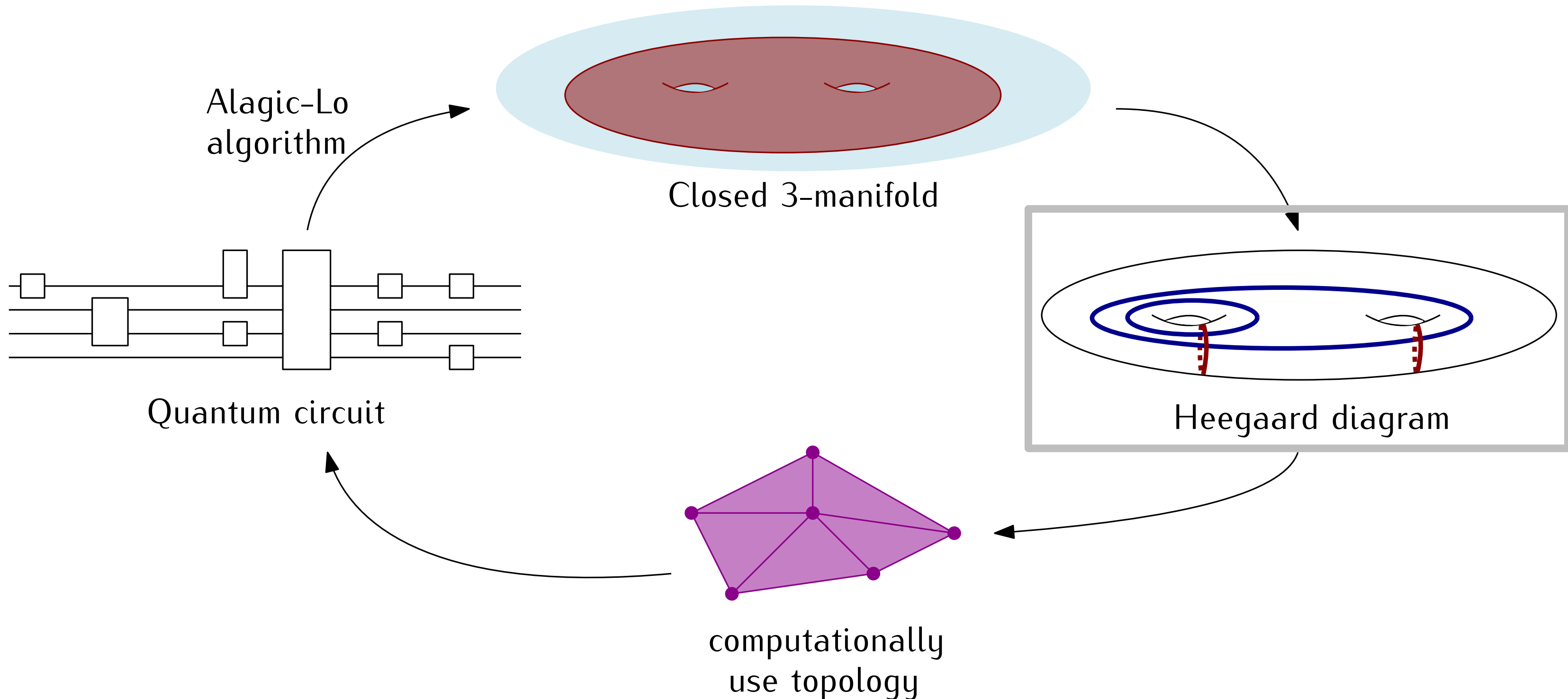
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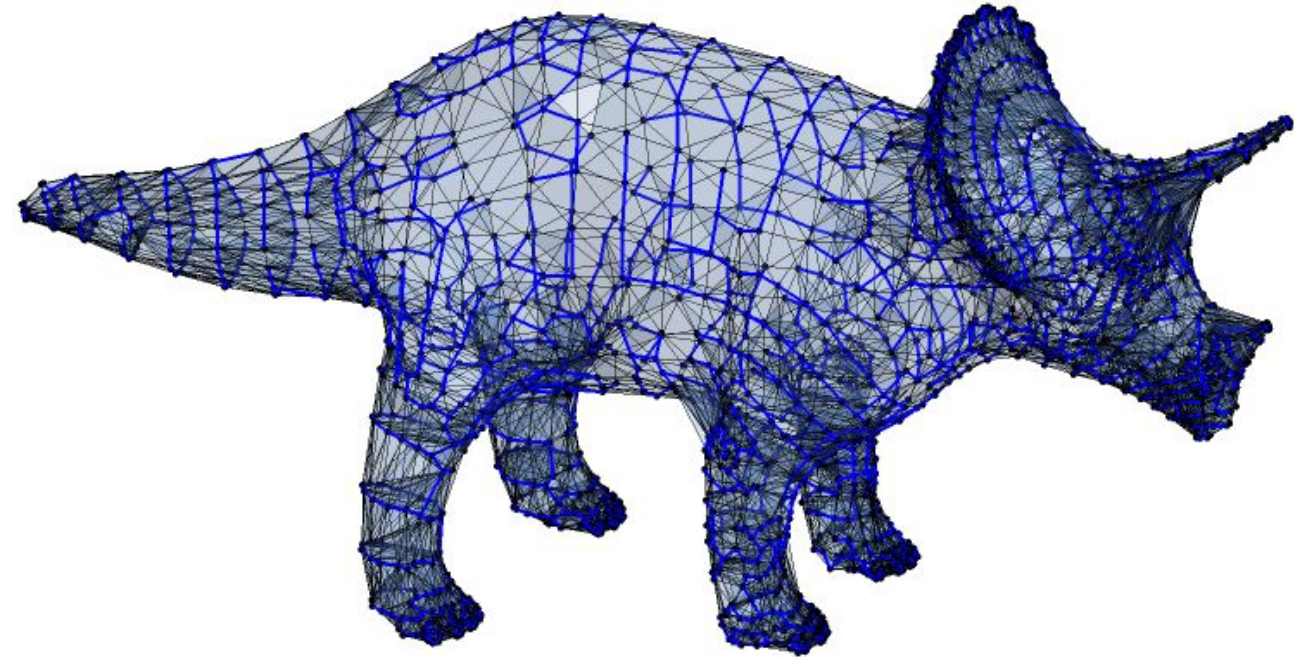
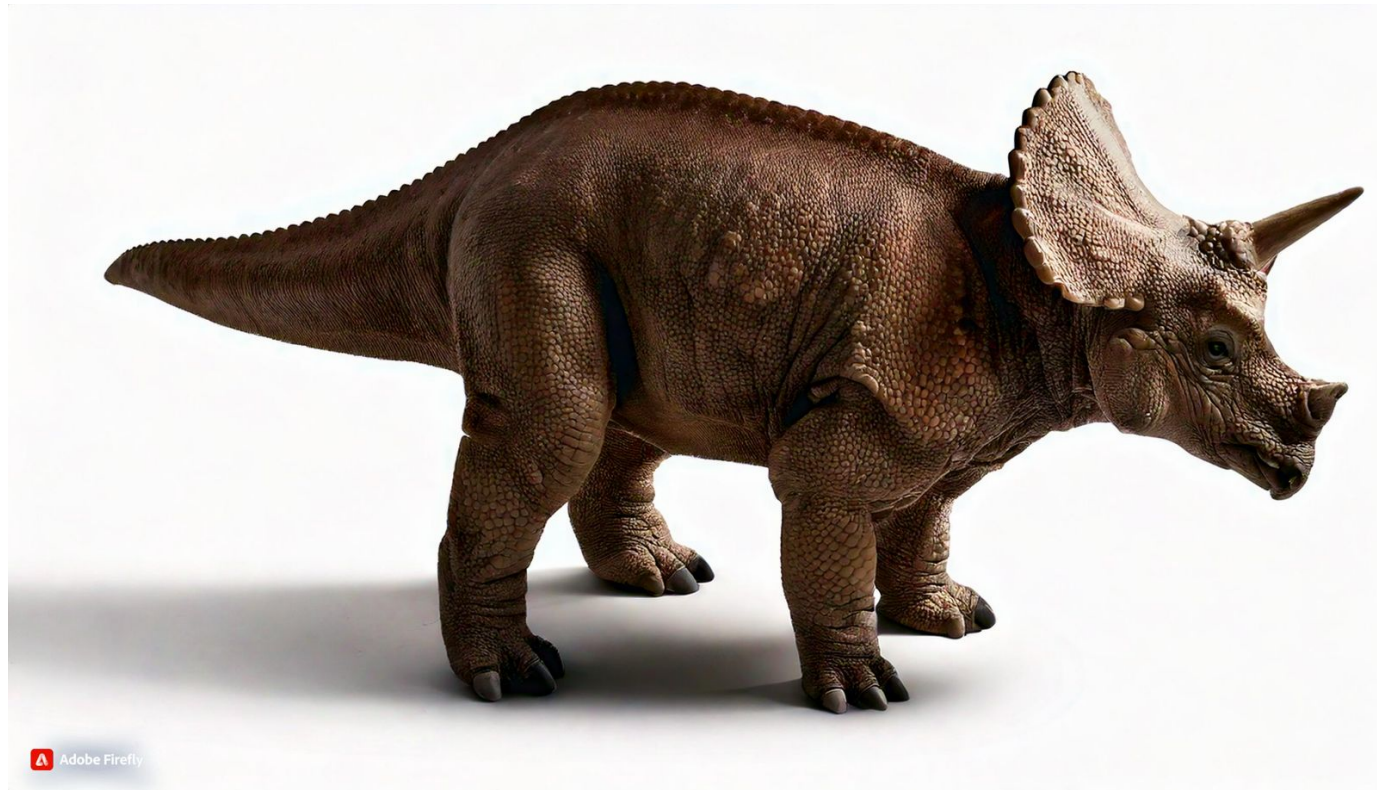
Conclusion

Roadmap: data structure for Heegaard diagrams



Triangulations of closed 3-manifolds

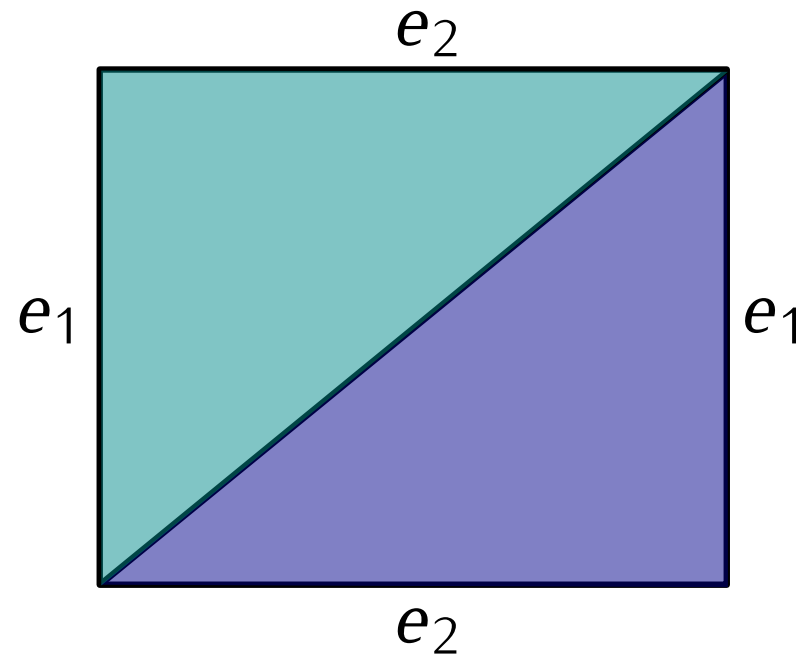
- Usually represent 3-manifolds by triangulations



Nice, but slightly inefficient in time and space....

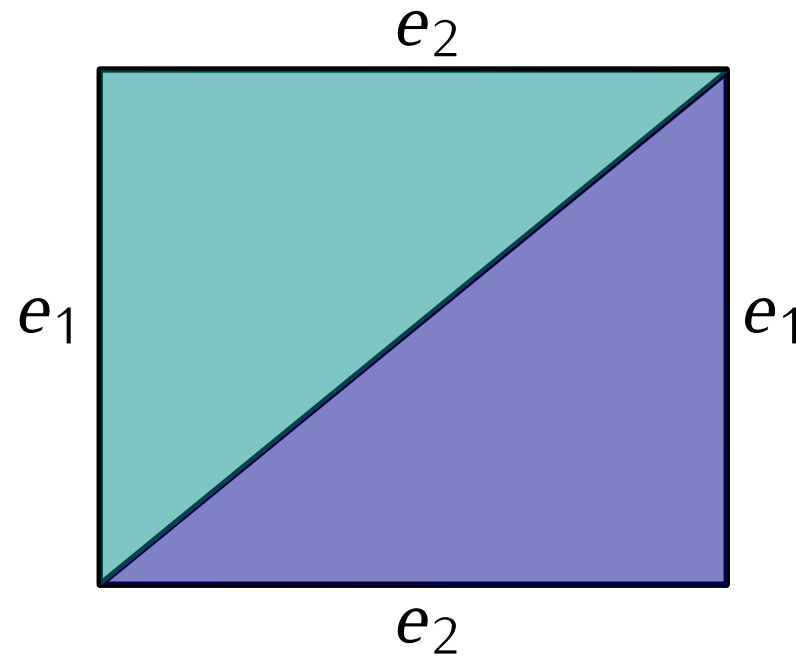
Our data structure for Heegaard splittings

- (Generalized) triangulation (of surface): graph with faces bounded by 3 edges

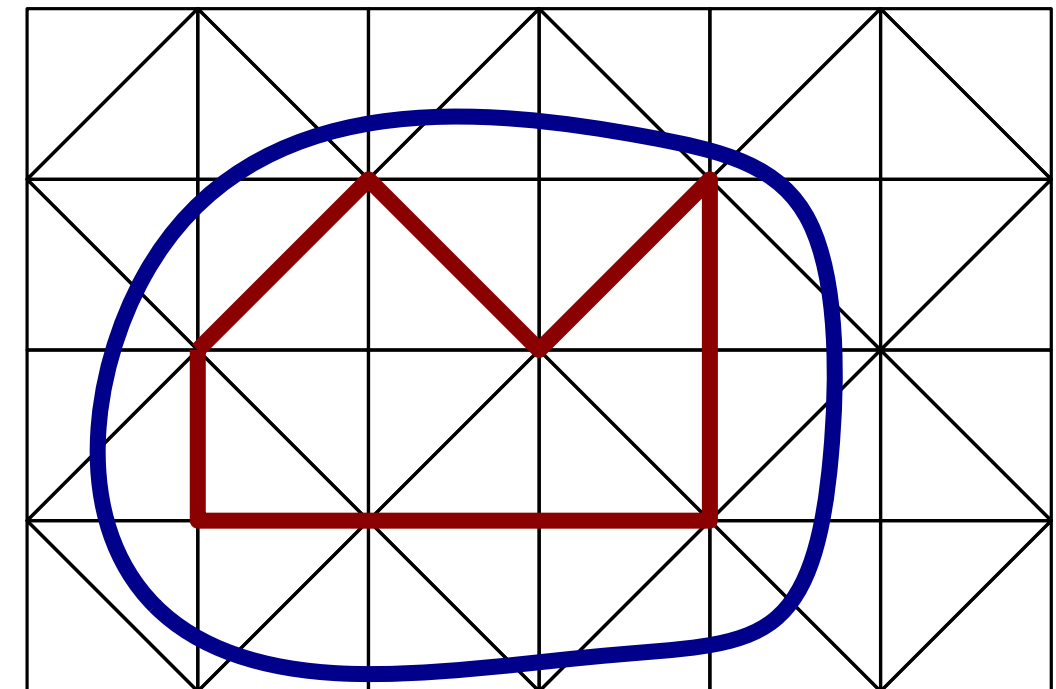


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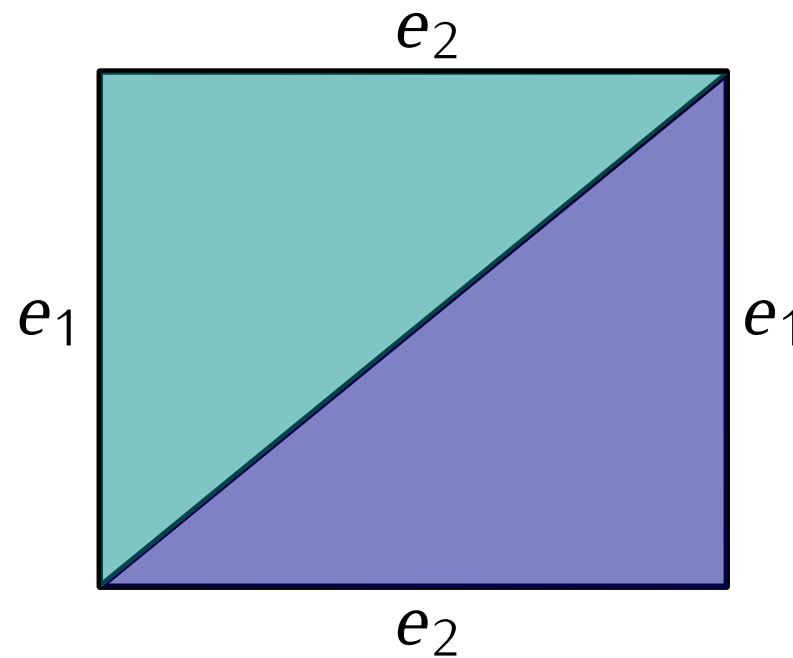


- A curve in a triangulation can be either
 - × **edged**: only lies on edges
 - × **normal**: transverse to the edges, given by the intersection **number** with each edge



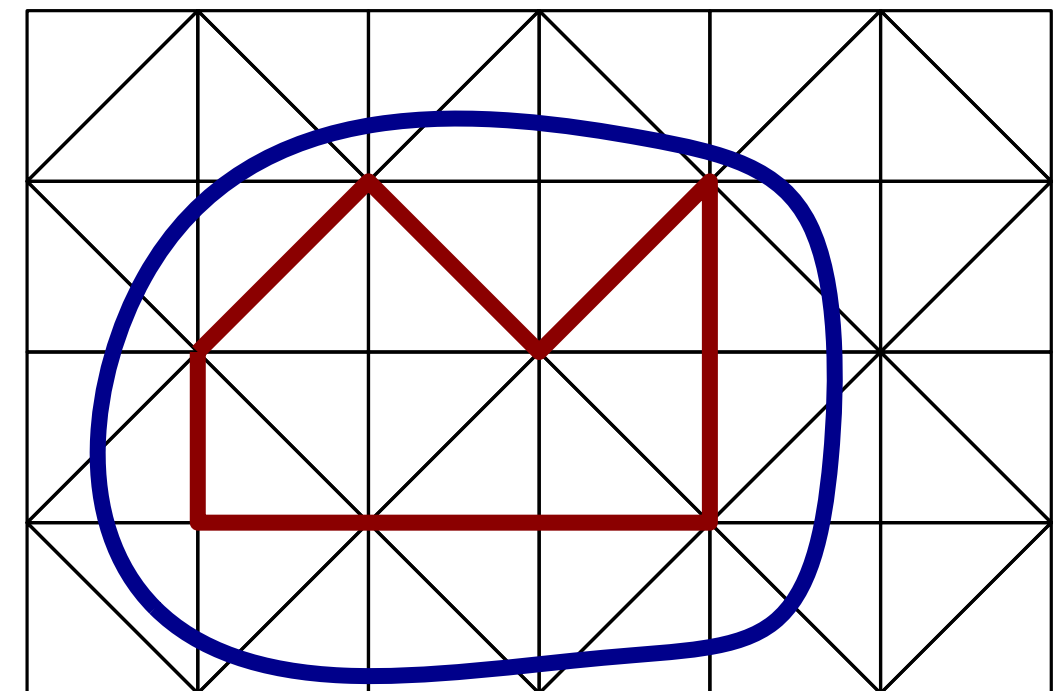
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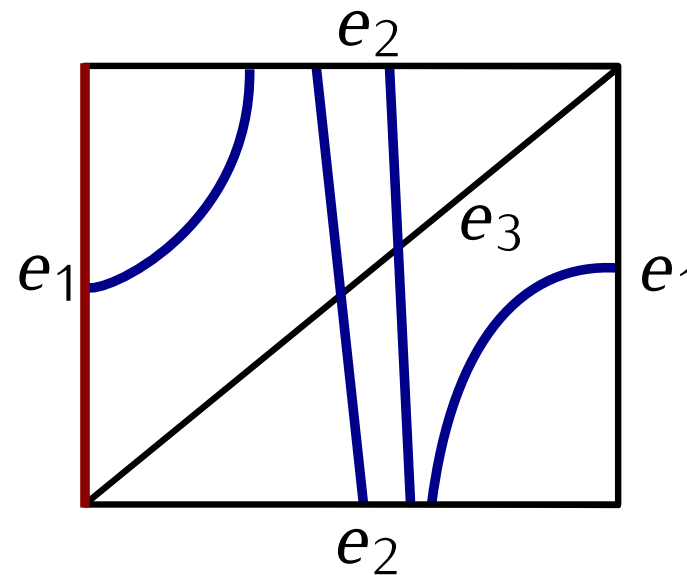
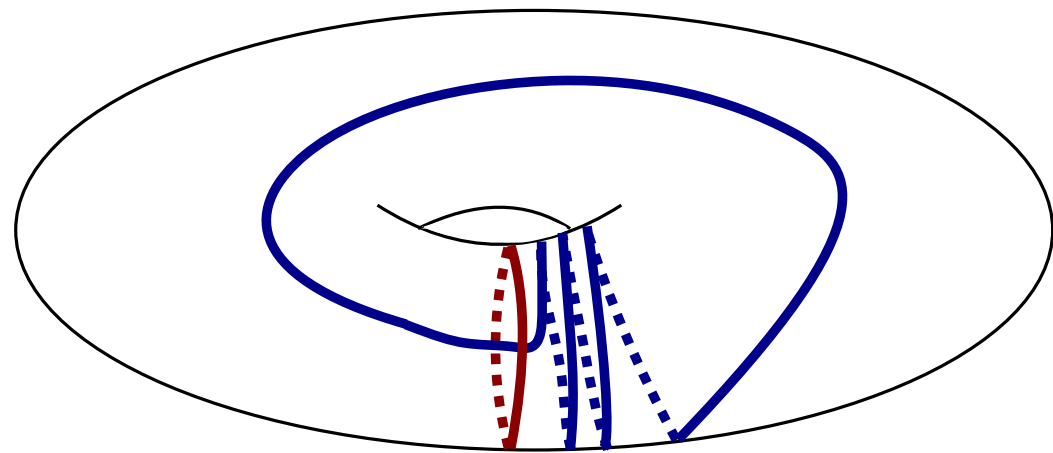
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→ vector of integers



Our data structure for Heegaard splittings

(Σ, α, β) encoded as $(T, E_T(\alpha), I_T(\beta))$



$$E_T(\alpha) = \{e_1\}$$
$$N_T(\beta) = \langle 1, 3, 2 \rangle$$

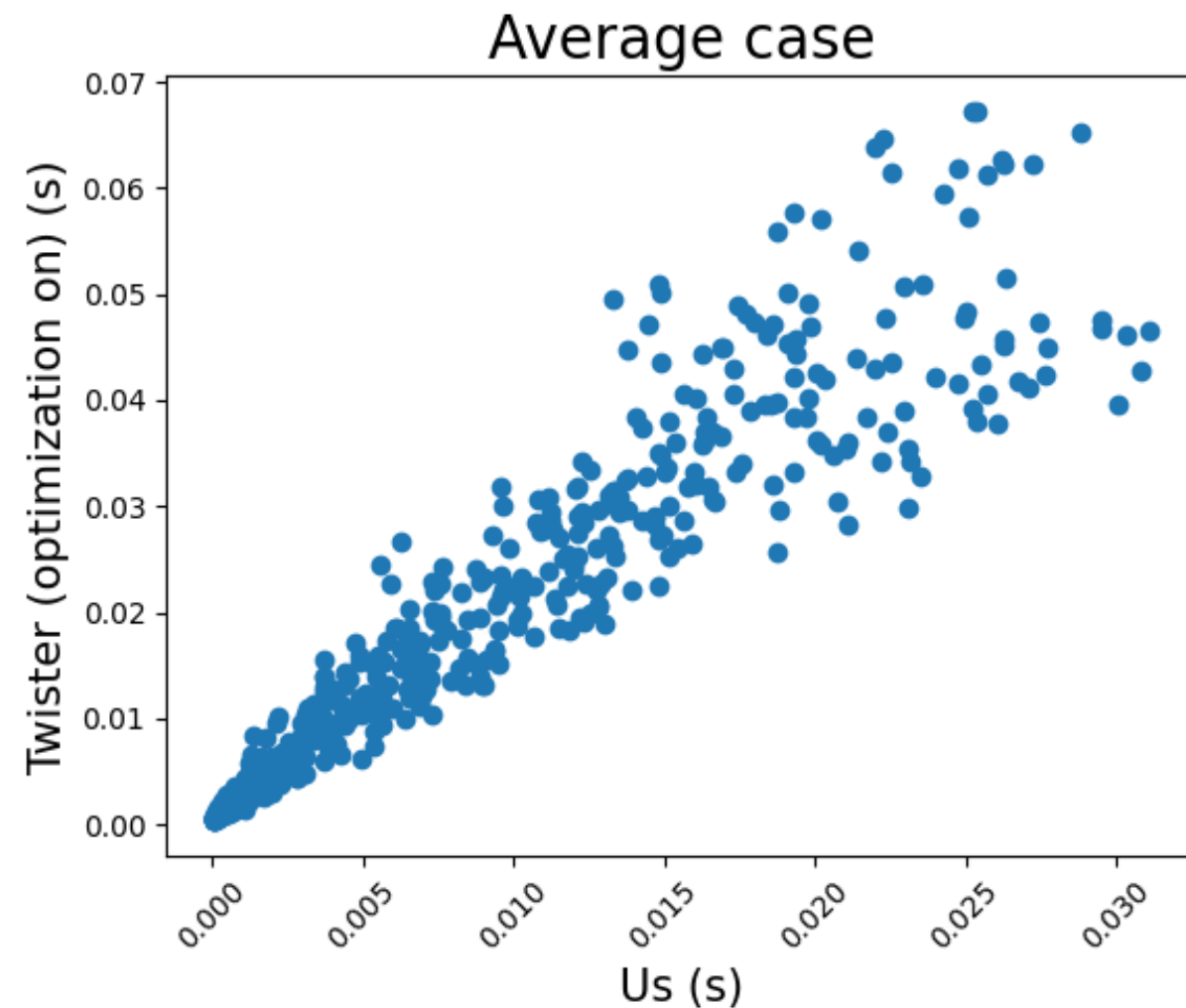
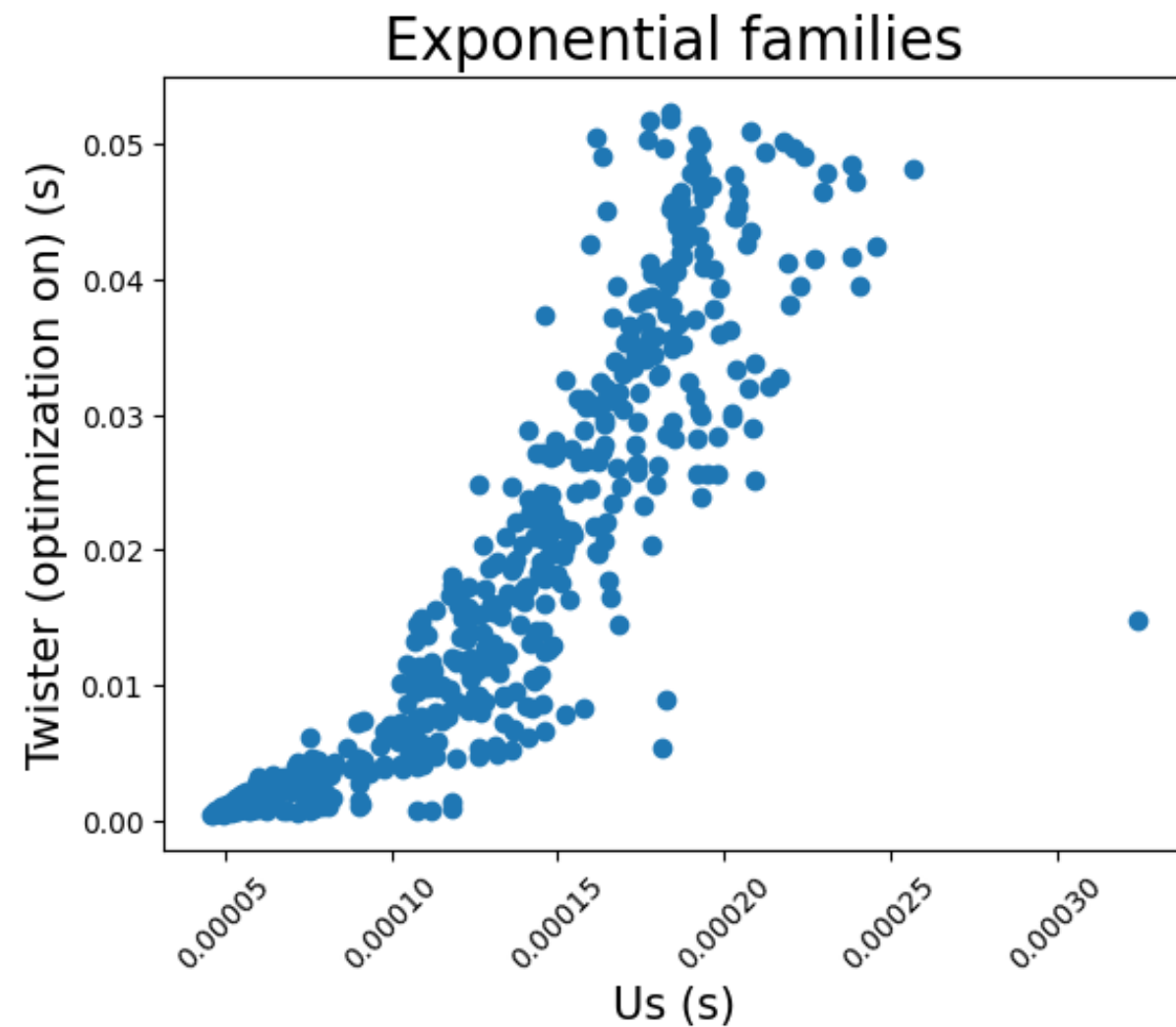
[HE & Maria, SoCG 2026] Theorem: there are families of closed 3-manifolds that require exponentially more space to triangulate than our structure

Several algorithms still efficient on this data structure

presentation of π_1 , presentation of H_1 , stabilizations, reducibility, ...

Experiments with our data structure

[HE & Maria, SoCG 2026] Solving problems on our structure are on average faster, sometimes exponentially



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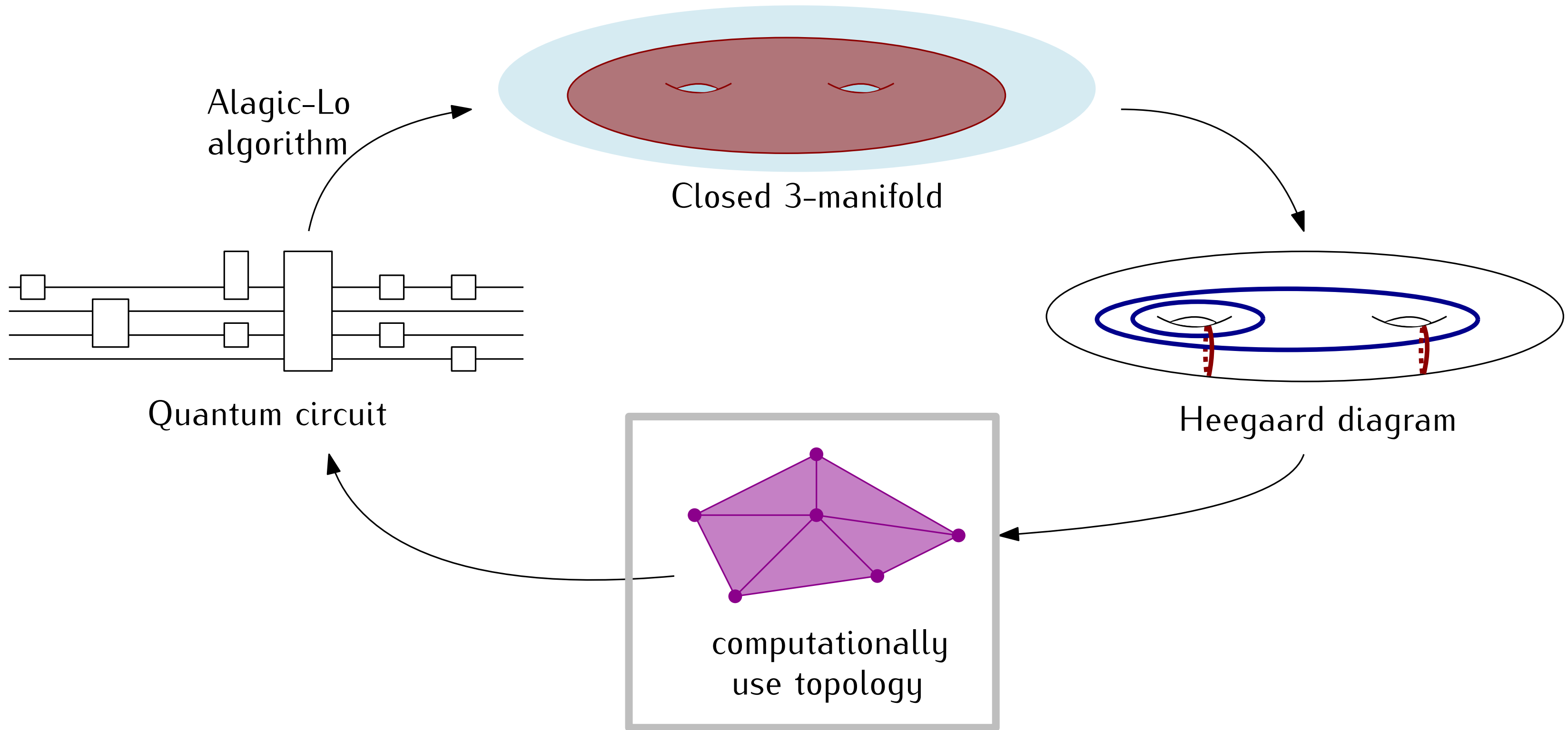
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⇒ Flags for error detection

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Roadmap: explore the topology



Does Alagic-Lo hold for special cases?

Does restricting the topology of M make the problem easier?

- **irreducible** $\approx$ if it is not equal to a connected sum $N_1 \# N_2$ where $N_1, N_2 \neq S^3$ [Gordon, 2021; Tencer, 2025]
- **hyperbolic** if it can be equipped with a complete hyperbolic metric [Thurston, 1982; Samperton, 2023]
- **small** $\approx$ if every small orientable embedded surface is compressible [Li, 2012]

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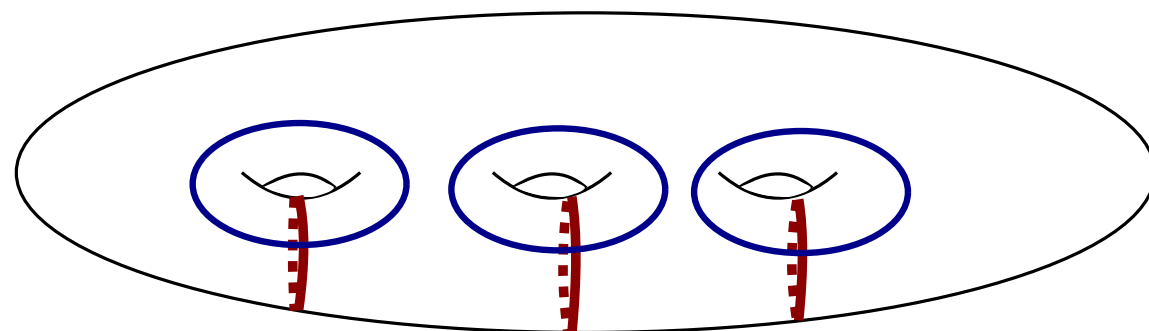
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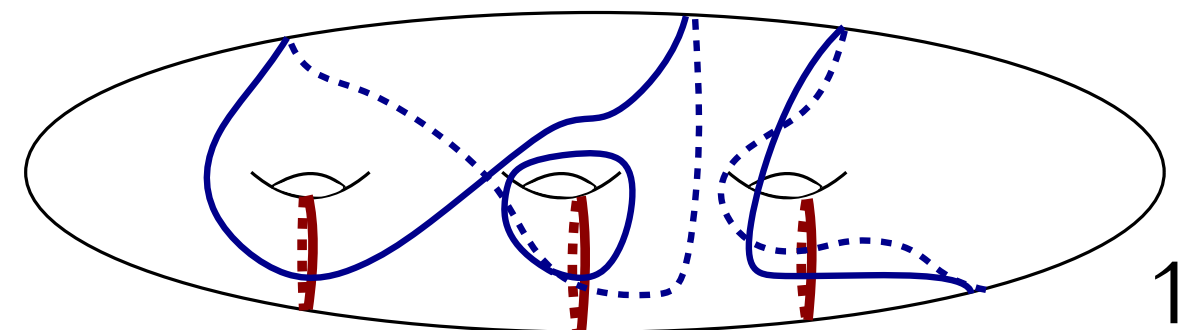
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Input diagram



polynomial time
invariant preserved

New diagram



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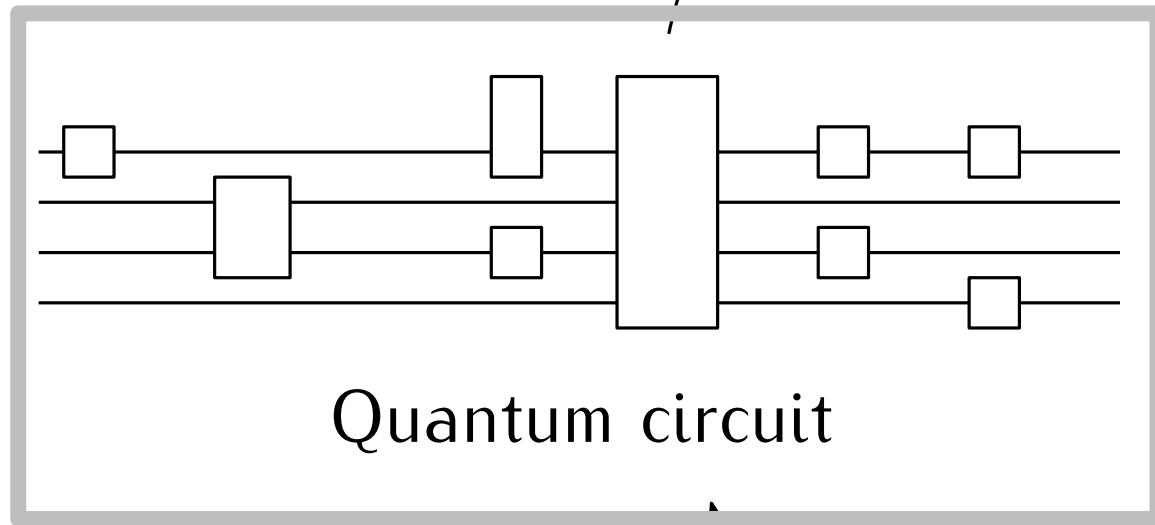
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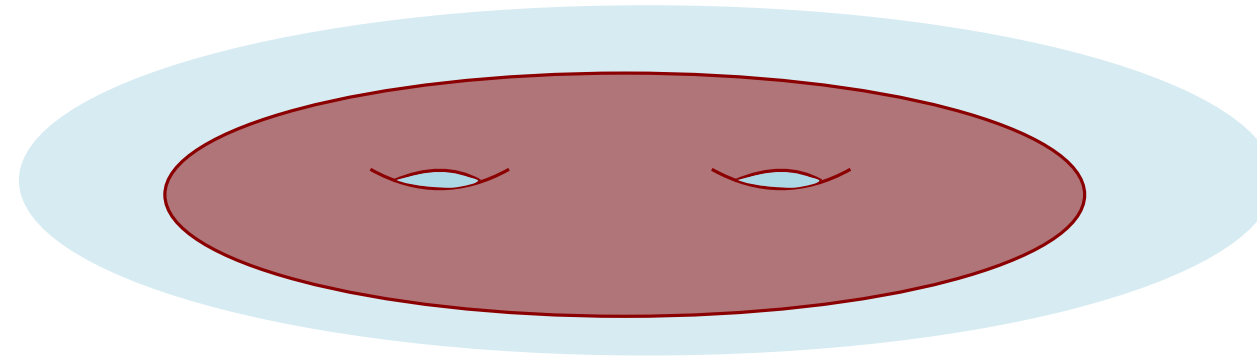
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Roadmap: quantum circuits

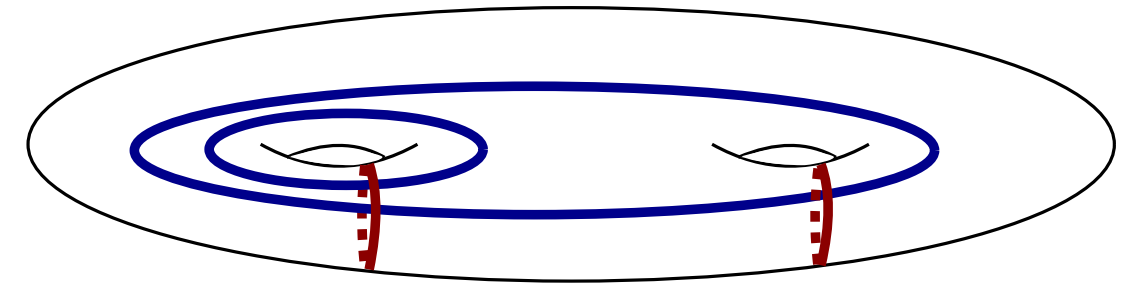
Alagic-Lo
algorithm



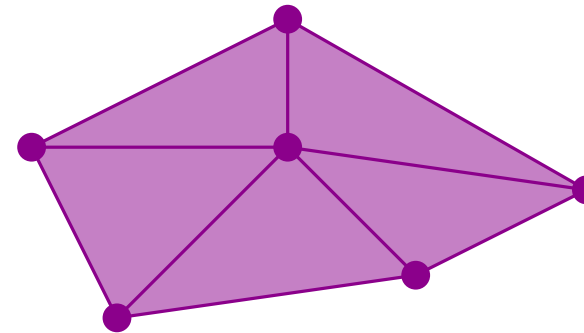
Quantum circuit



Closed 3-manifold



Heegaard diagram



computationally
use topology

Quantum mechanics

- **Qudit:** quantum physical system that is a normalized vector in $\mathbb{C}^d$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A general qudit is a linear combination $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ with

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

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- The time-evolution of a qudit is described by multiplying through a special unitary matrix

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

Quantum mechanics

- A state $|\psi\rangle$ can be **measured** along a computational states and the probability of measuring it as $|\varphi\rangle$ is

$$\text{Pr}(\varphi) = |\langle\varphi|\psi\rangle|^2$$

Ex.: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$

$$\text{Pr}(0) = |\langle 0|\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \right|^2 = |\alpha|^2$$

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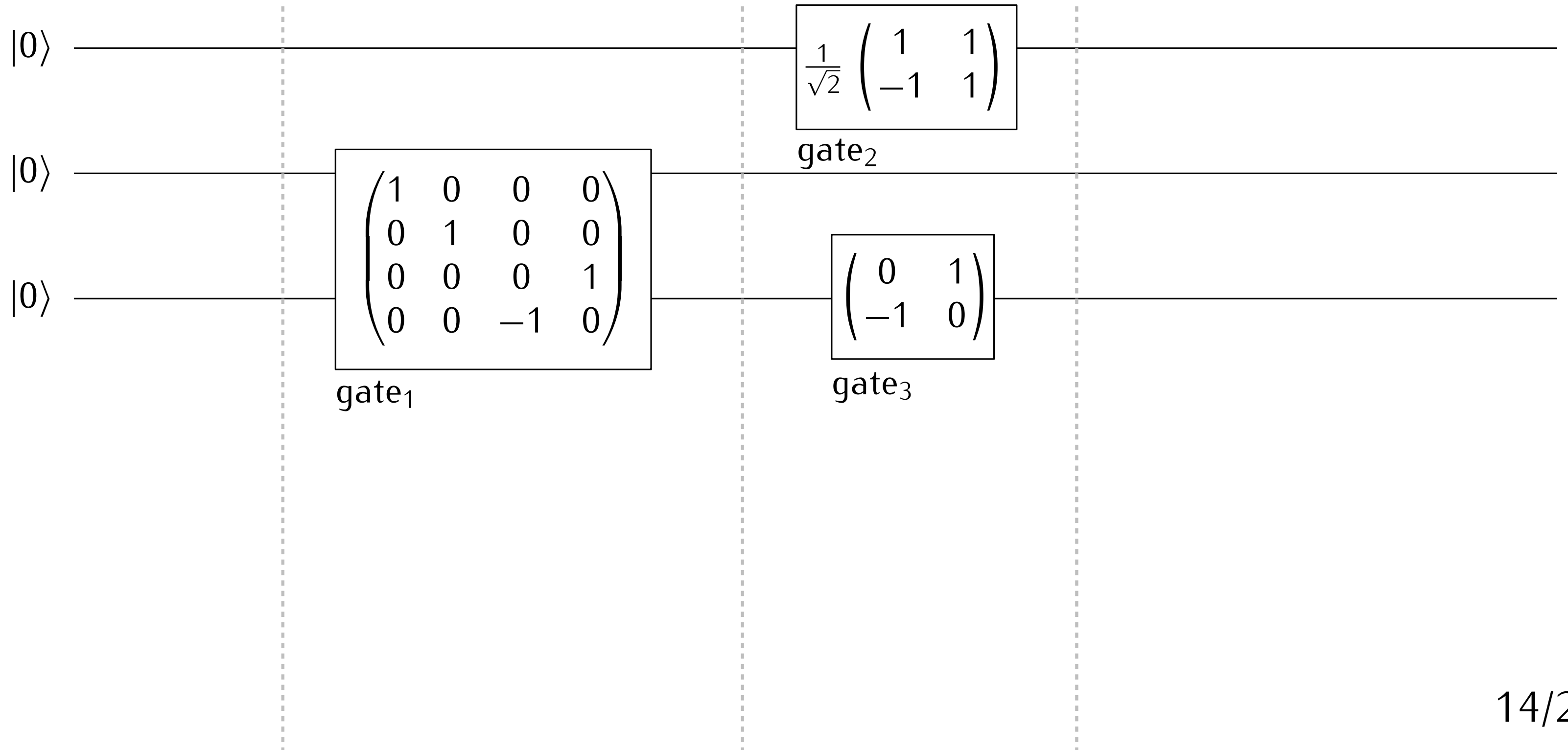
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- If $|\psi\rangle$ and $|\phi\rangle$ are two qudits, their overall state is given by their **tensor product**

The dimension of a system of n qudits is d^n

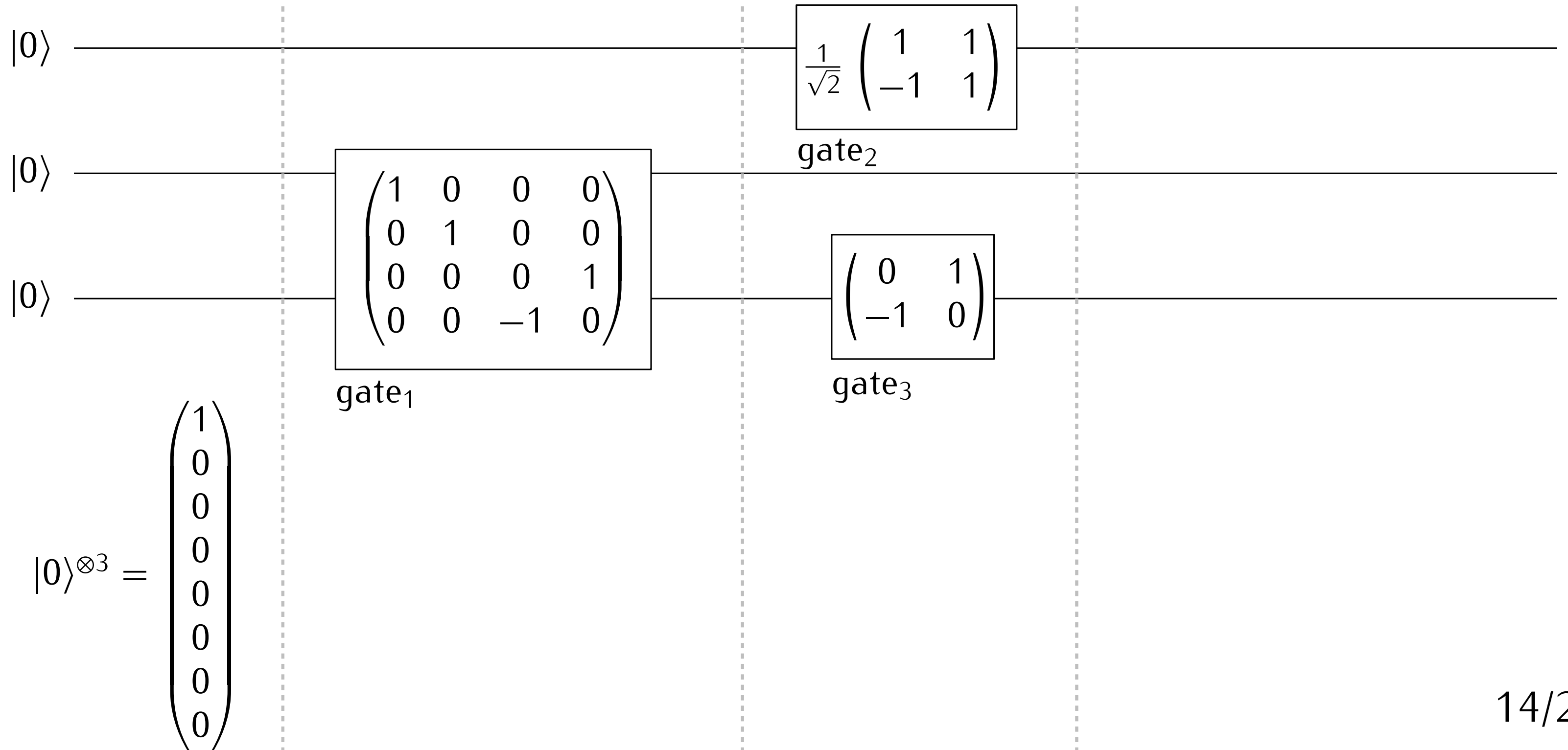
Quantum circuits

- Quantum circuits explore qudit rules for computations ($d = 2$)



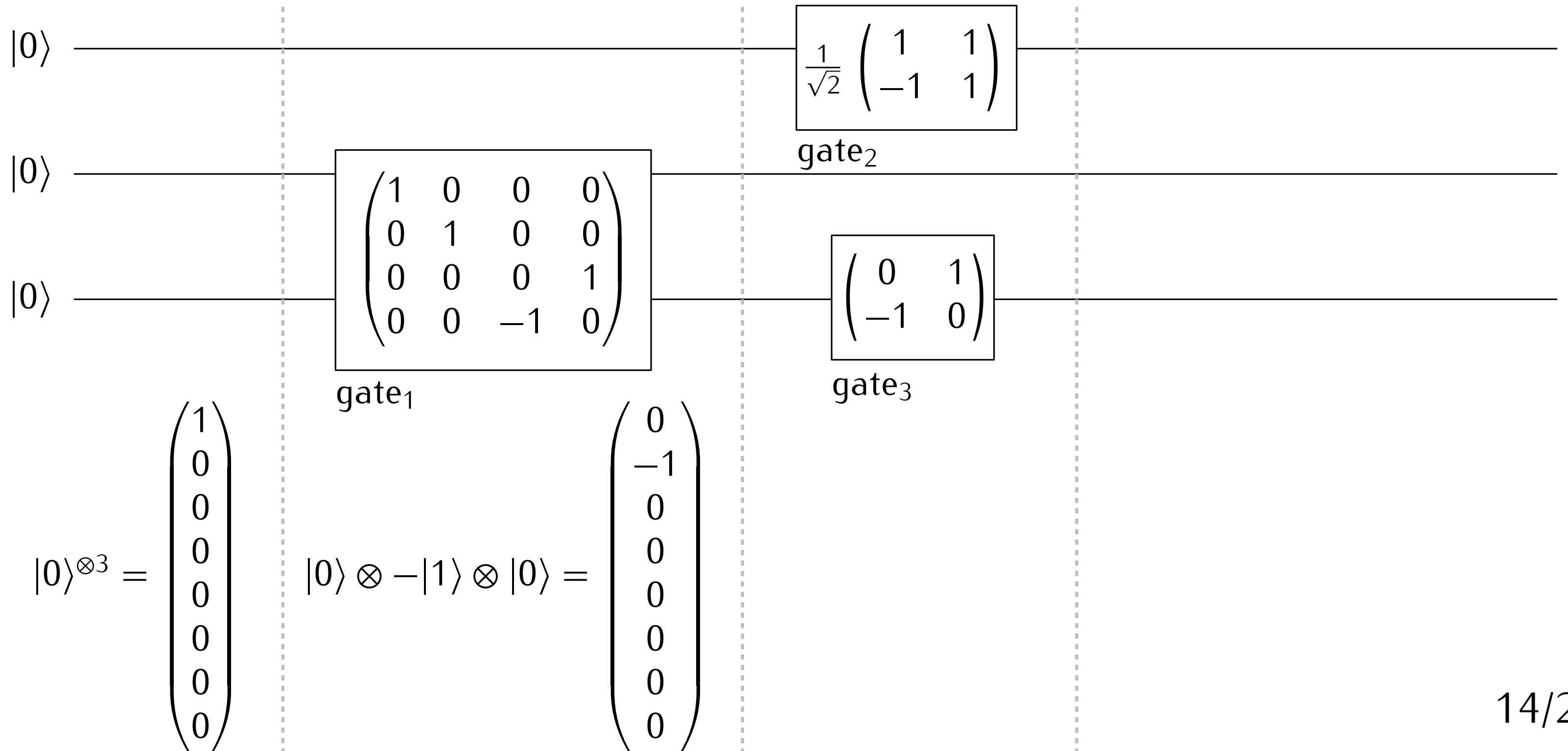
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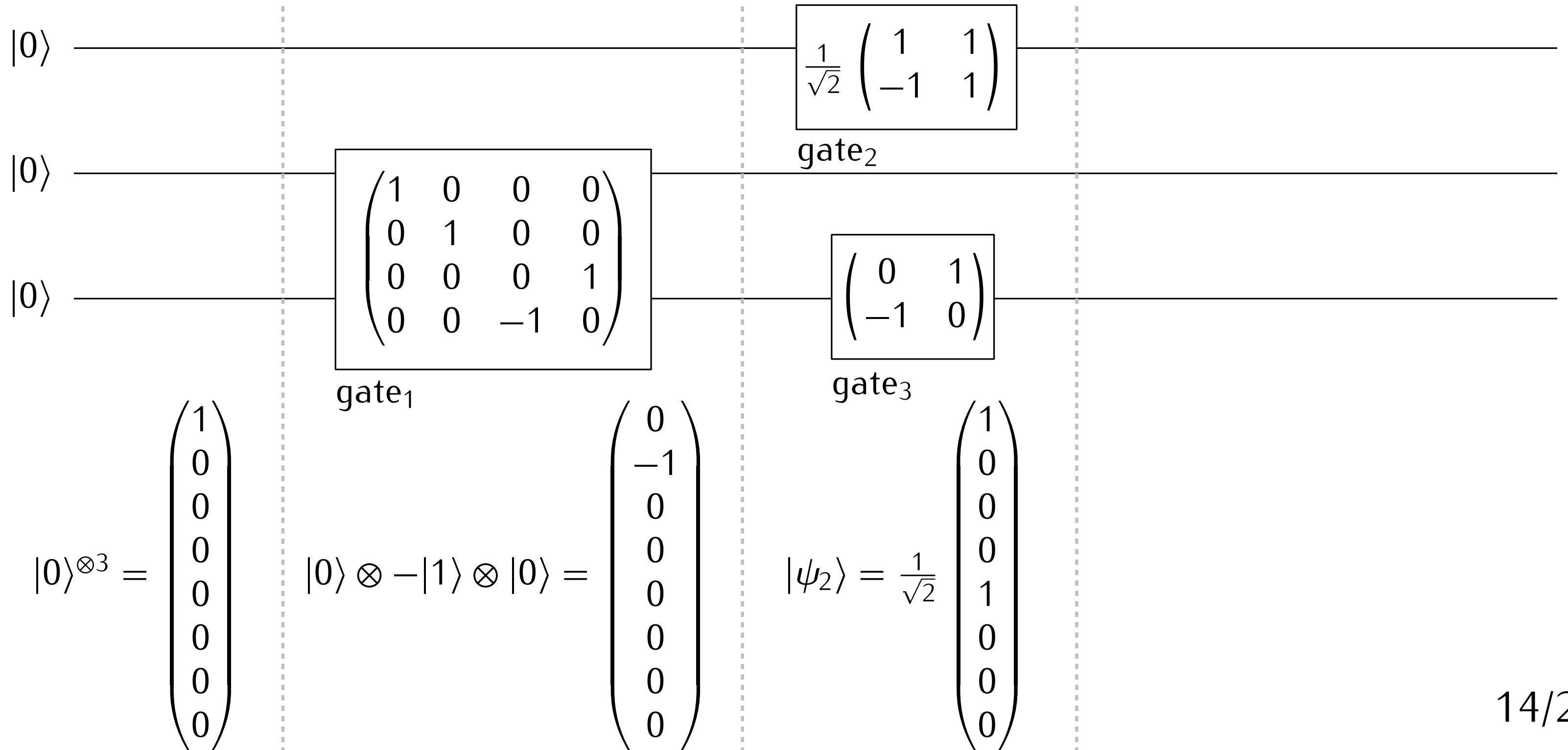
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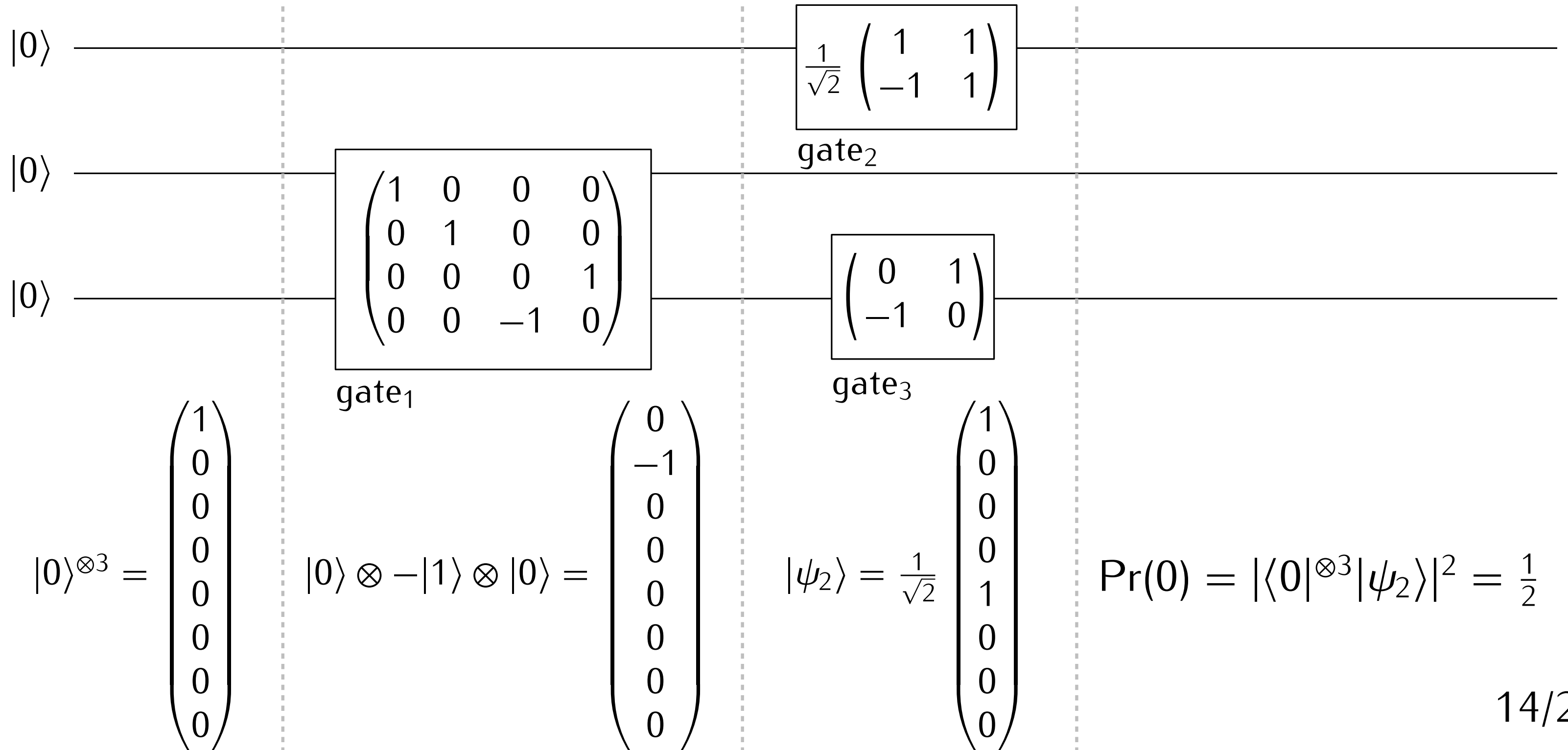
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Solovay-Kitaev algorithm

- Can approximate any $d^n \times d^n$ unitary to arbitrary precision

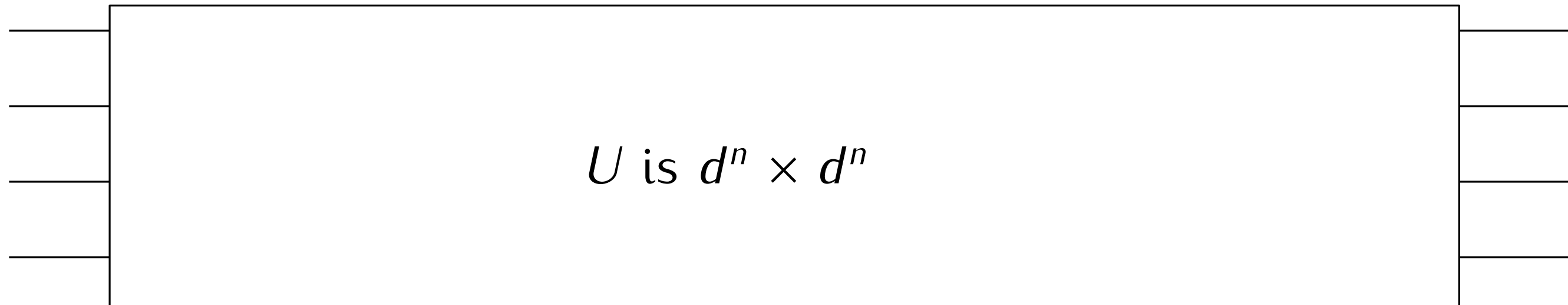
Solovay-Kitaev theorem: a universal set of **implementable** gates can be used to very efficiently approximate any small matrix

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Usual quantum computing problem



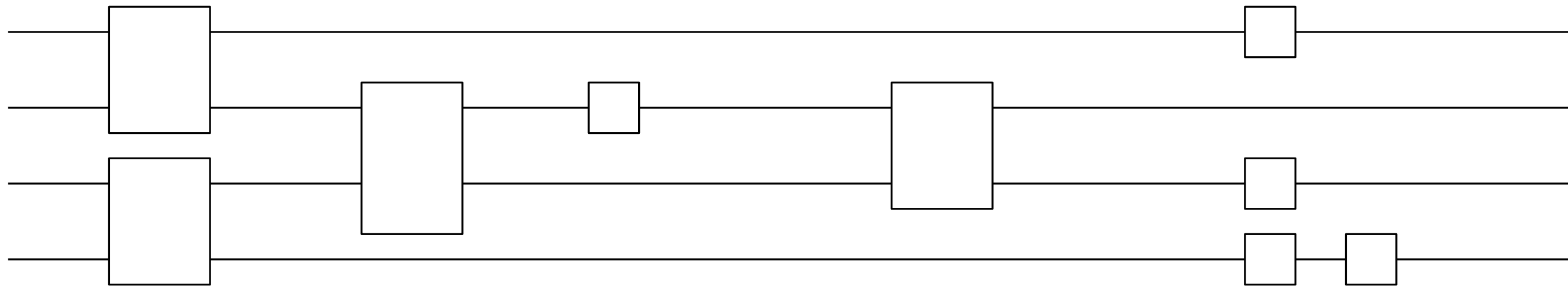
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Solovay-Kitaev theorem: a universal set of **implementable** gates can be used to very efficiently approximate any small matrix

Usual quantum computing problem

1. [Muthukrishnan & Stroud Jr, 2000] break U into small matrices



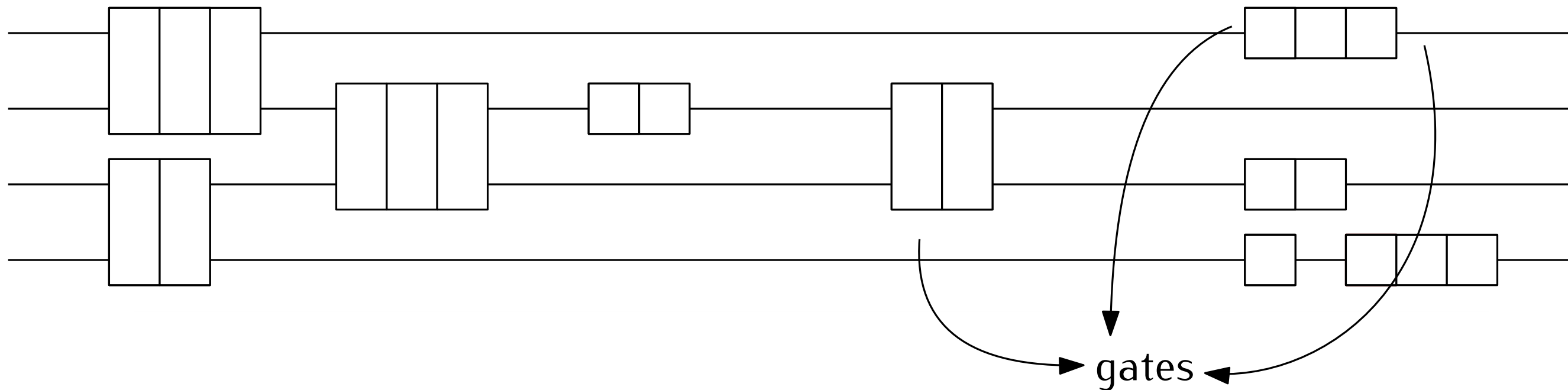
Solovay-Kitaev algorithm

- Can approximate any $d^n \times d^n$ unitary to arbitrary precision

Solovay-Kitaev theorem: a universal set of **implementable** gates can be used to very efficiently approximate any small matrix

Usual quantum computing problem

1. [Muthukrishnan & Stroud Jr, 2000] break U into small matrices
2. approximate each piece with gates using SK



Introduction

Part 1: Topology

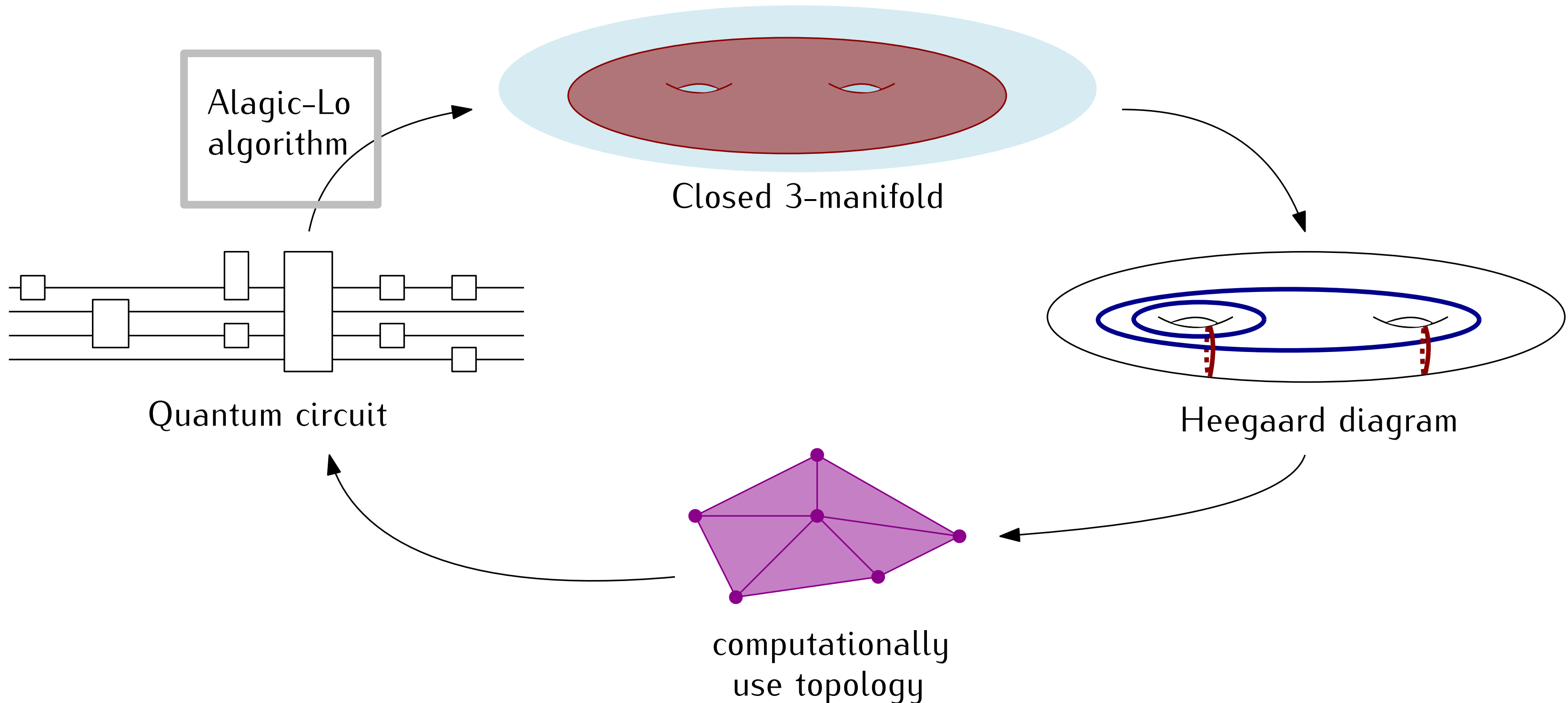
- Heegaard splittings
- ⇒ Data structure for Heegaard splittings
- ⇒ Restricting manifold topology

Part 2: Quantum computing

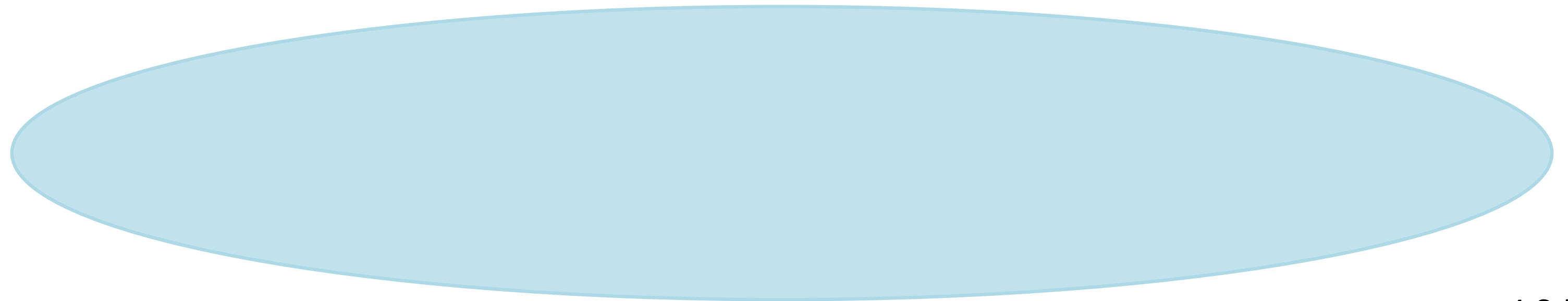
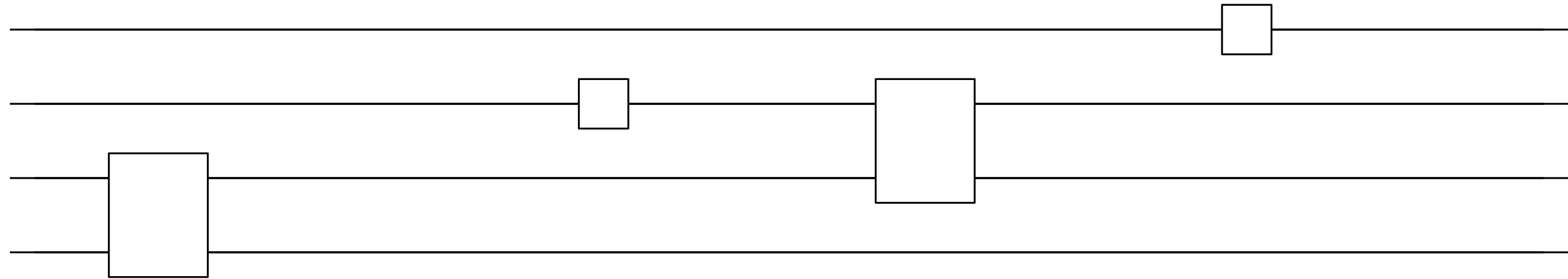
- Quantum circuits
- ⇒ Geometric Solovay-Kitaev
- ⇒ Flags for error detection

Conclusion

Roadmap: map circuits to manifolds

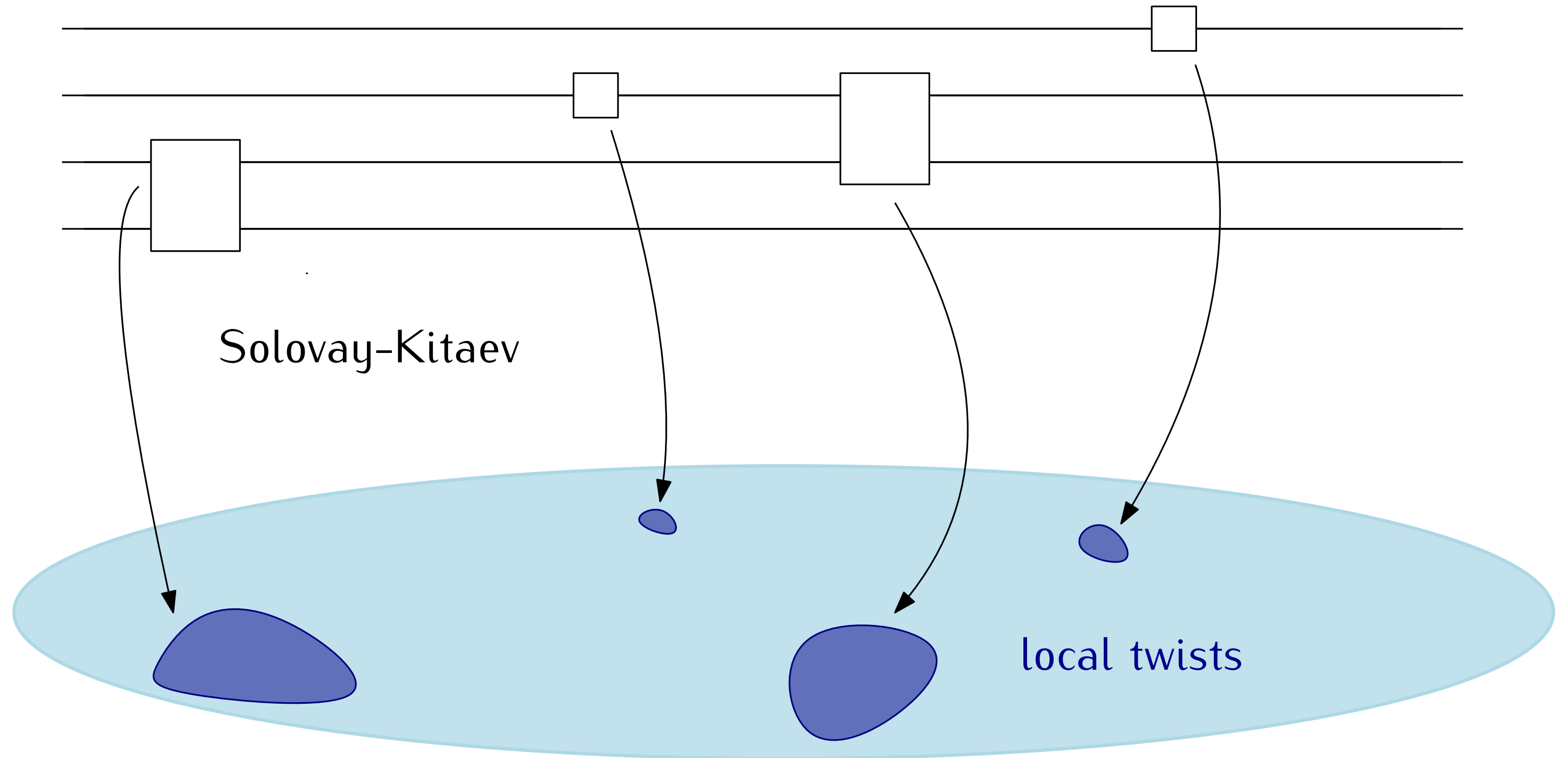


Implementing Alagic-Lo



Fixed manifold: S^3

Implementing Alagic-Lo



Fixed manifold: S^3

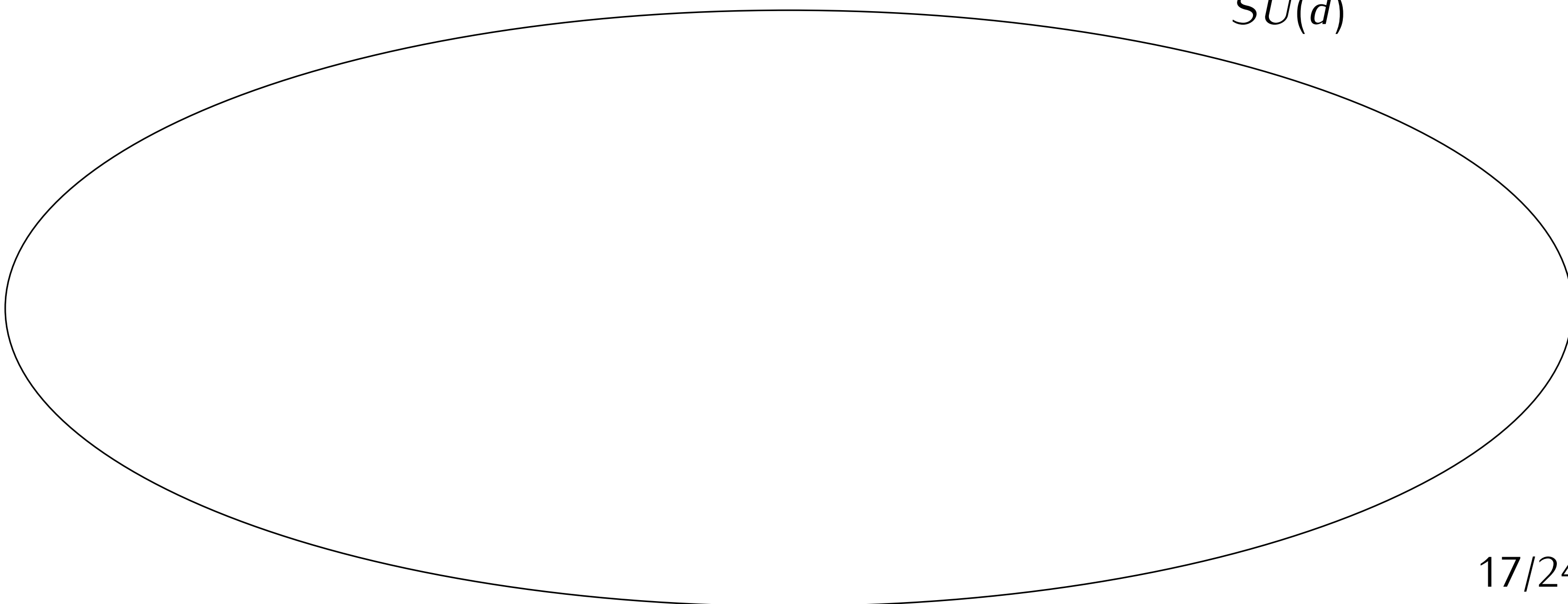
The Solovay-Kitaev algorithm

SK Theorem: time and local twists scale as $\log^c(1/\epsilon)$, **BUT** exponential in d

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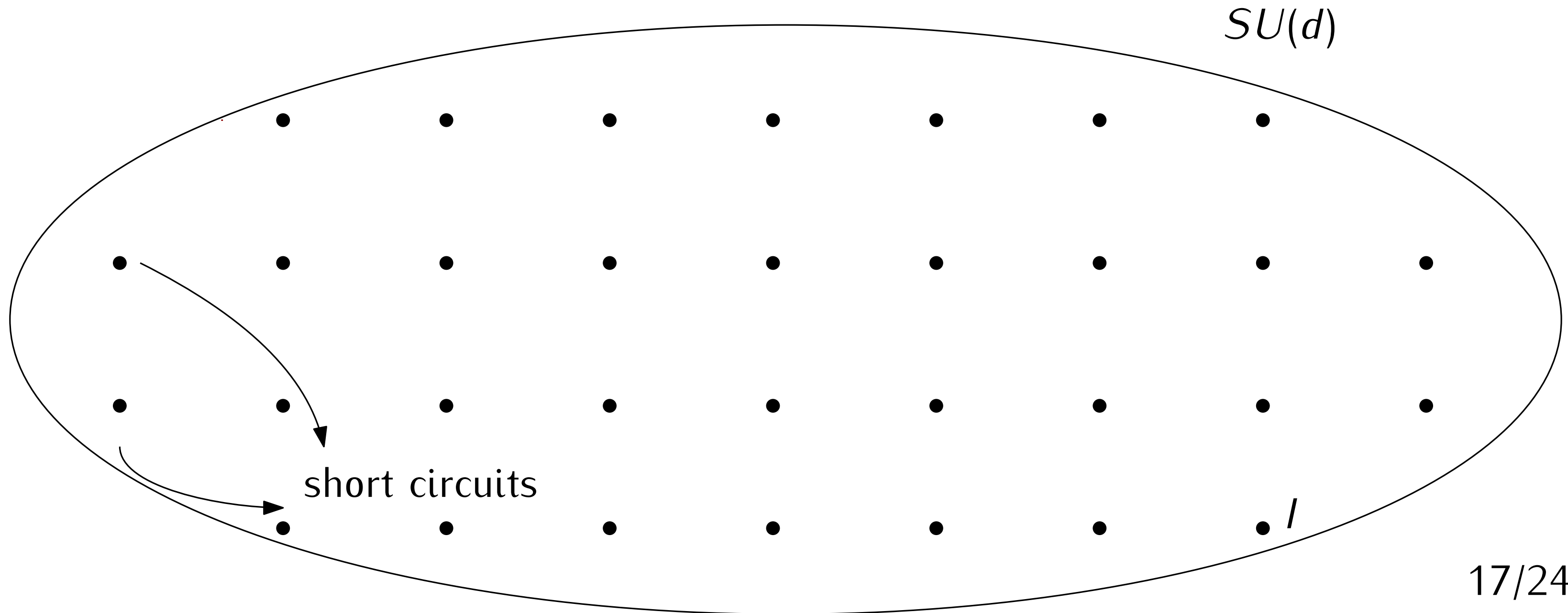
$SU(d)$



The Solovay-Kitaev algorithm

SK Theorem: time and local twists scale as $\log^c(1/\epsilon)$, **BUT** exponential in d

SK starts with manually enumerating short circuits

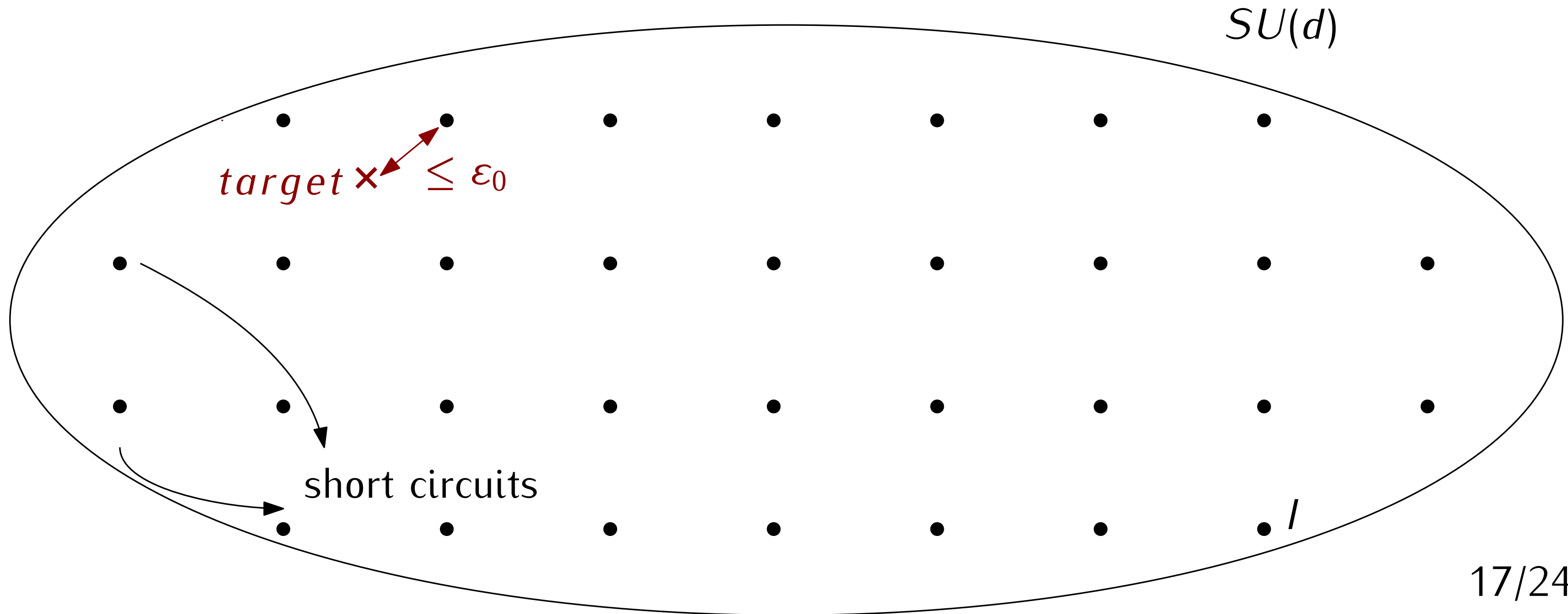


The Solovay-Kitaev algorithm

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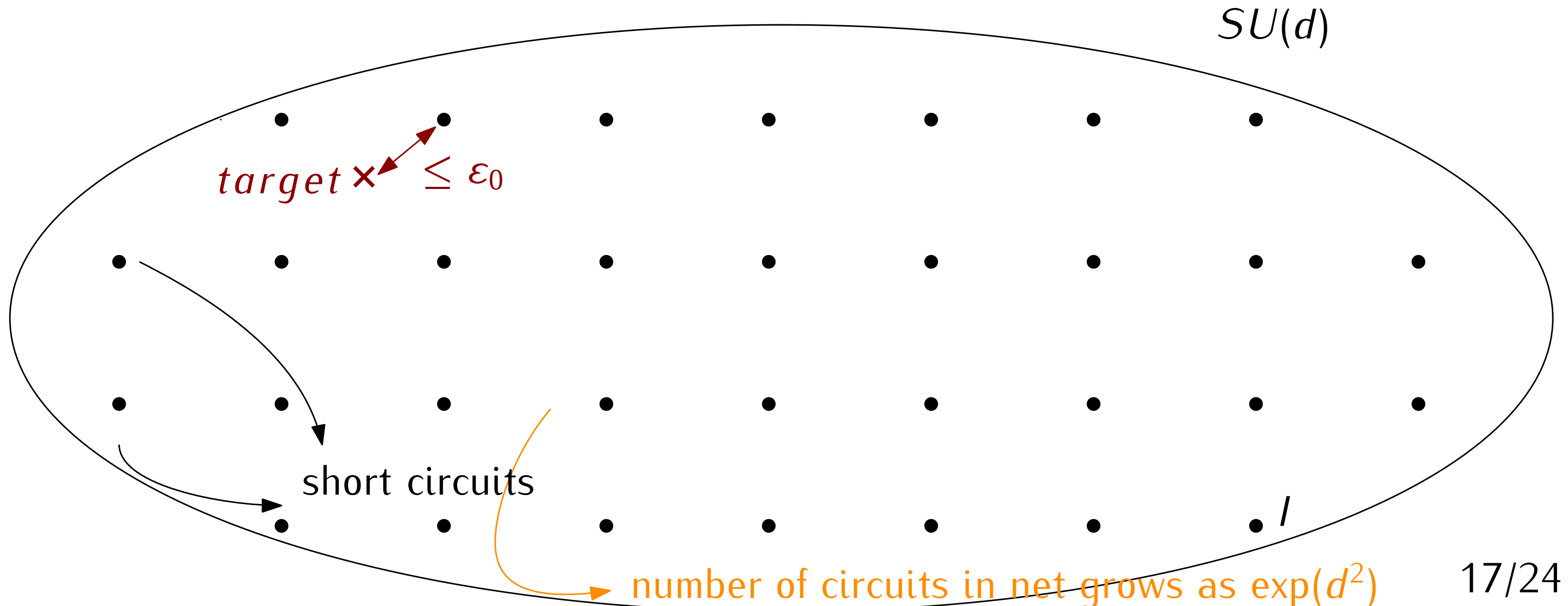


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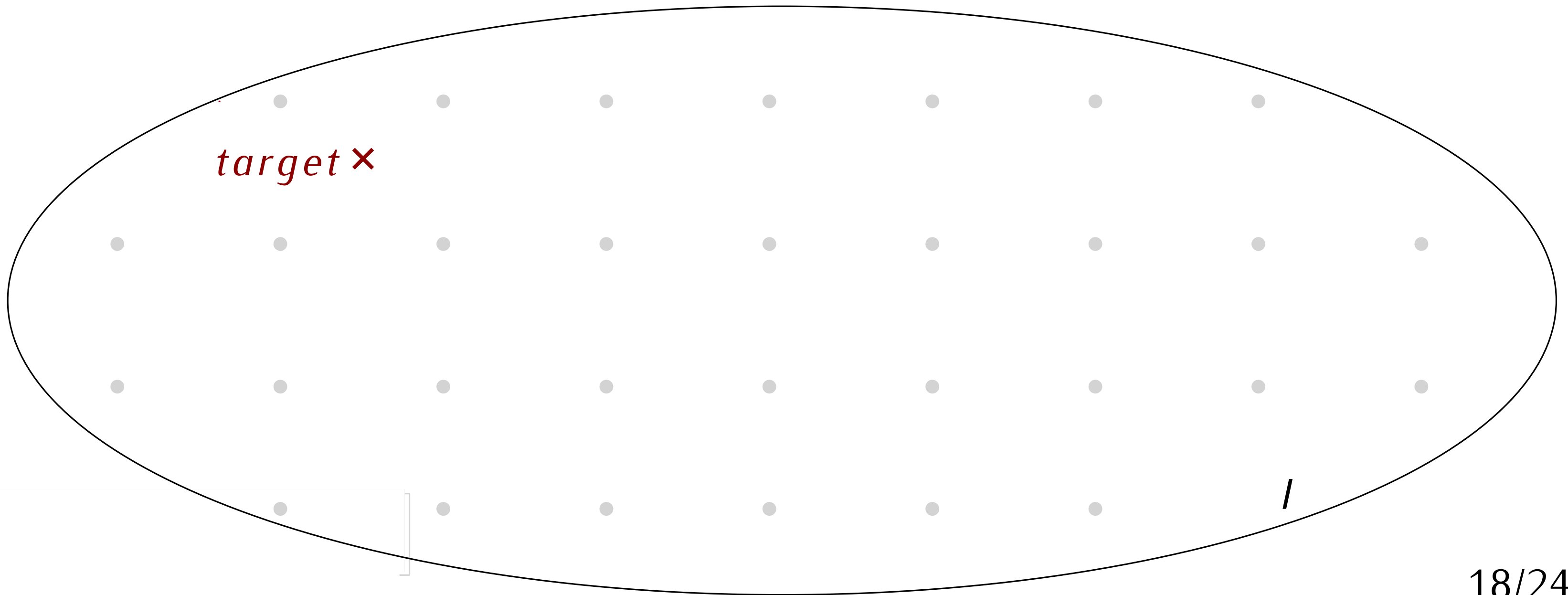
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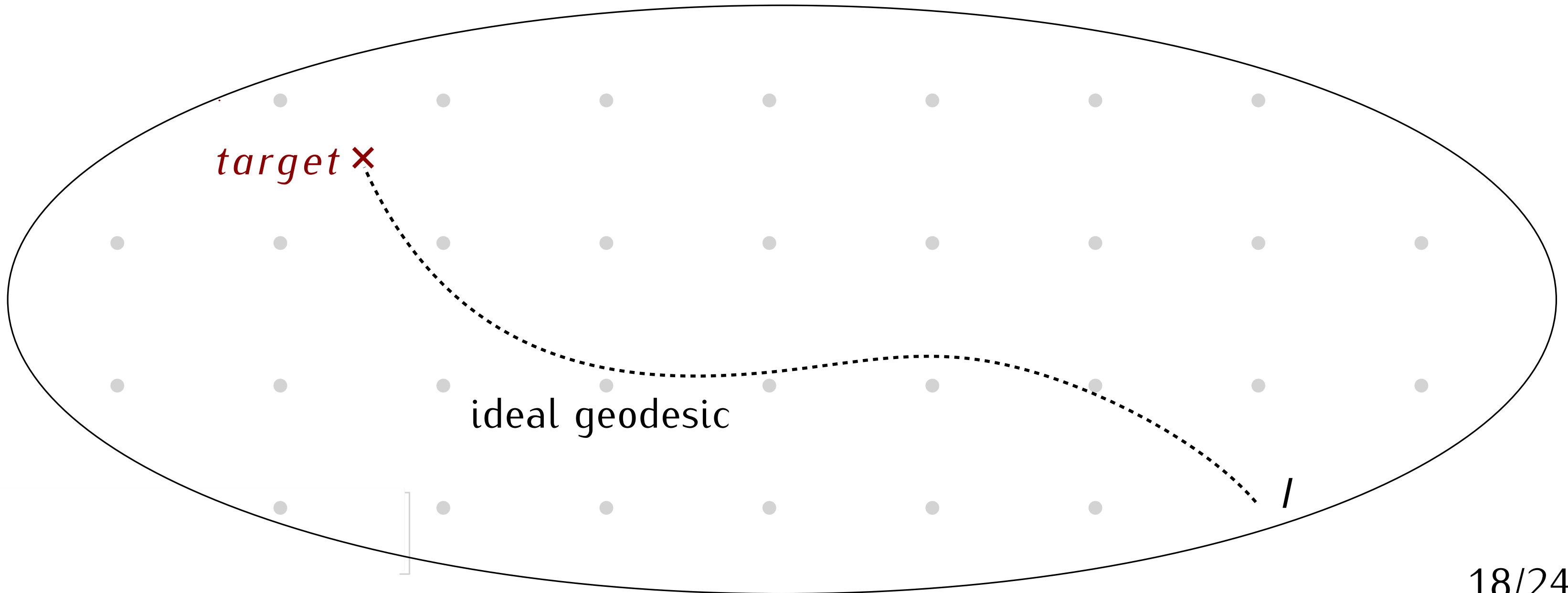


Our modification to Solovay-Kitaev



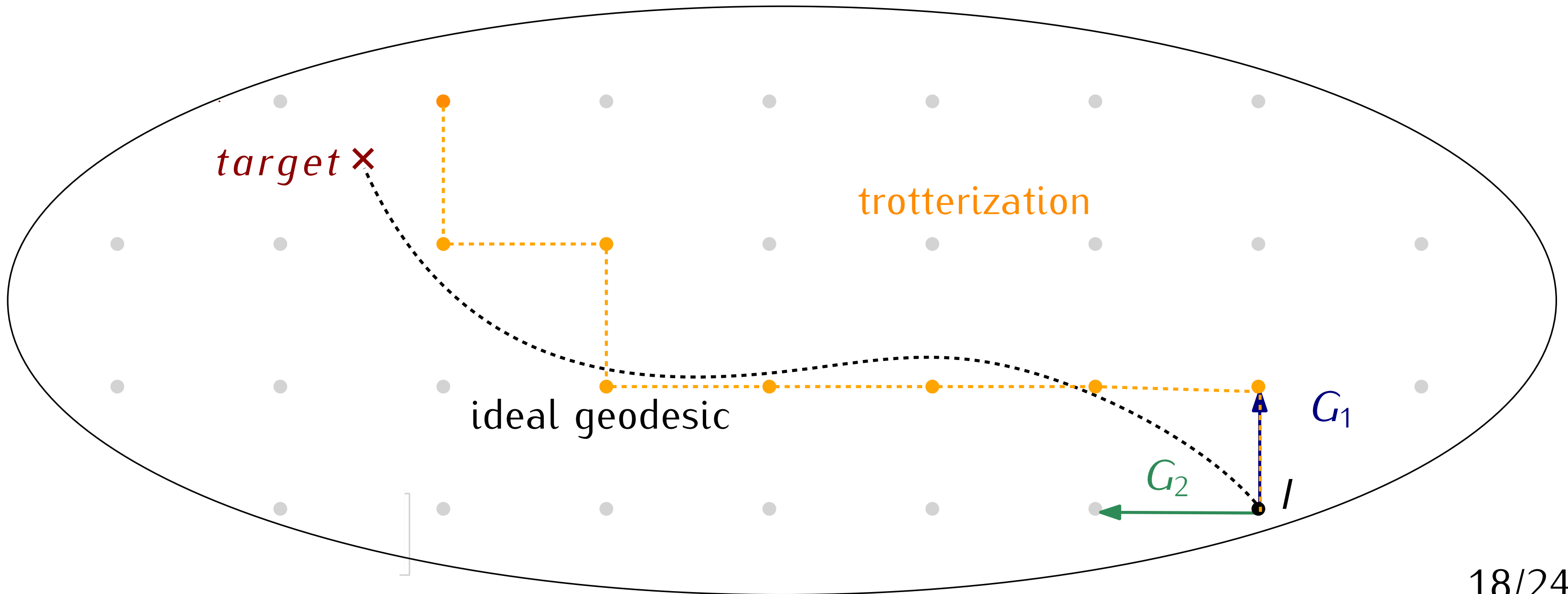
Our modification to Solovay-Kitaev

- **Idea:** “integrate” geodesic using small gates



Our modification to Solovay-Kitaev

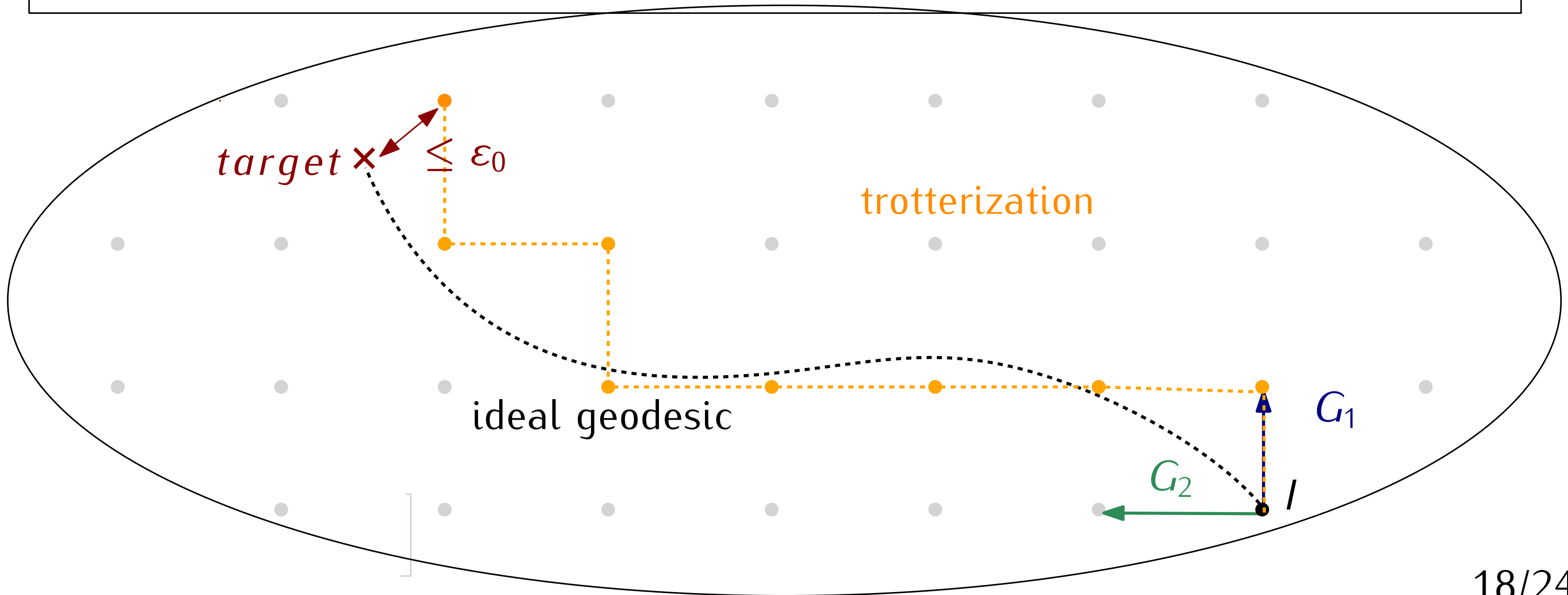
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Our modification to Solovay-Kitaev

- **Idea:** “integrate” geodesic using small gates

Use trotterization to substitute the net search subroutine, then follow usual SK



Our modification to Solovay–Kitaev

- [SK] $O(\exp(d^2) \log^c(1/\varepsilon))$
- [HE & Maria (good case), submitted to SODA] $O(\text{poly}(d) \log^c(1/\varepsilon))$
- [HE & Maria (general case), submitted to SODA] $O(\exp(d^2) + \text{poly}(d) \log^c(1/\varepsilon))$

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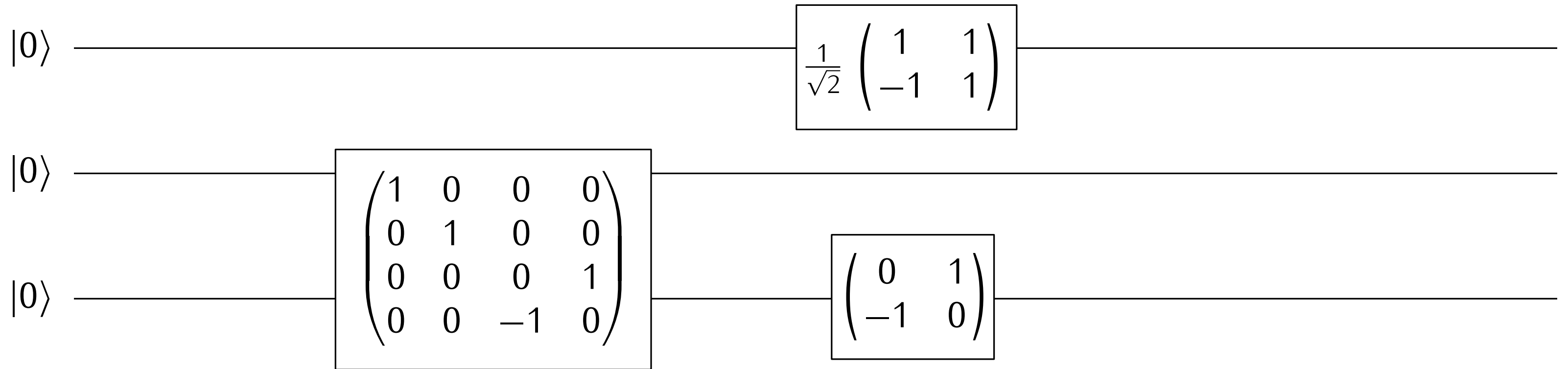
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- $\implies$ Geometric Solovay-Kitaev
- $\implies$ Flags for error detection

Conclusion

What about other types of errors?

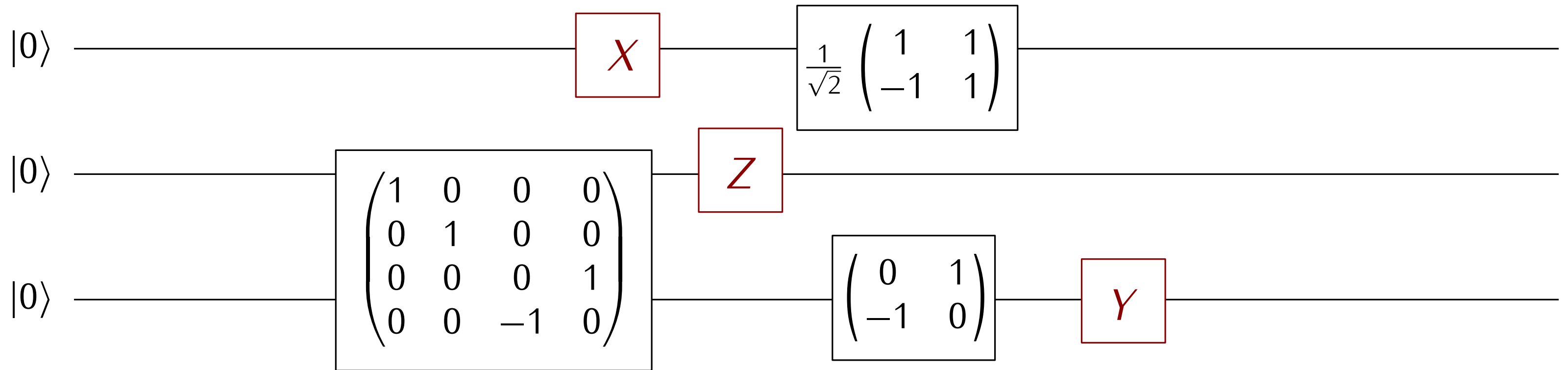
- SK suppresses implementation error, but there are others



- One of them is the unwanted application of **Pauli matrices** to the individual qubits

What about other types of errors?

- SK suppresses implementation error, but there are others

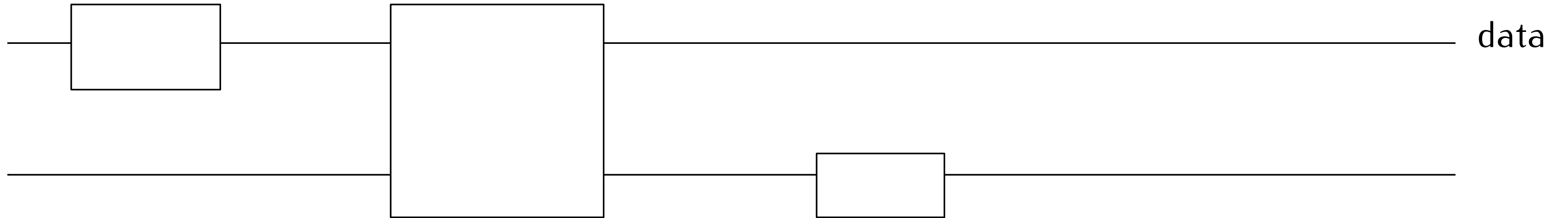


- One of them is the unwanted application of **Pauli matrices** to the individual qubits

[Knill & Laflamme, 1996]: Pauli errors are a natural starting point to more general noise models

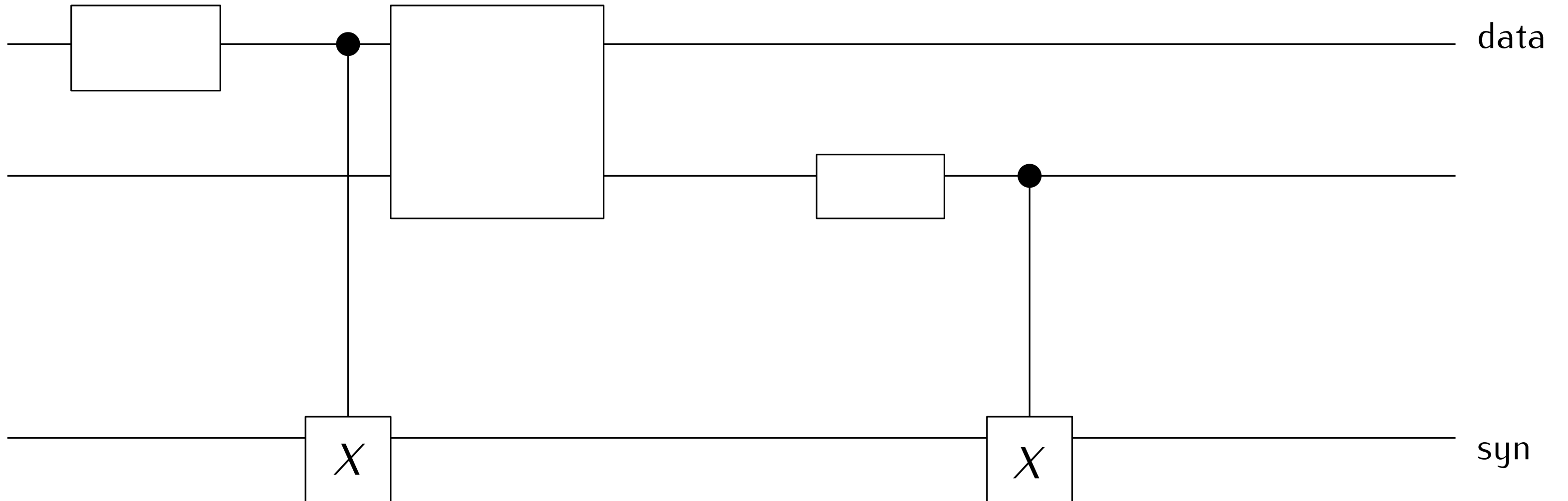
Error detection

- The usual way to **detect** Pauli errors is to use **syndrome measurements**



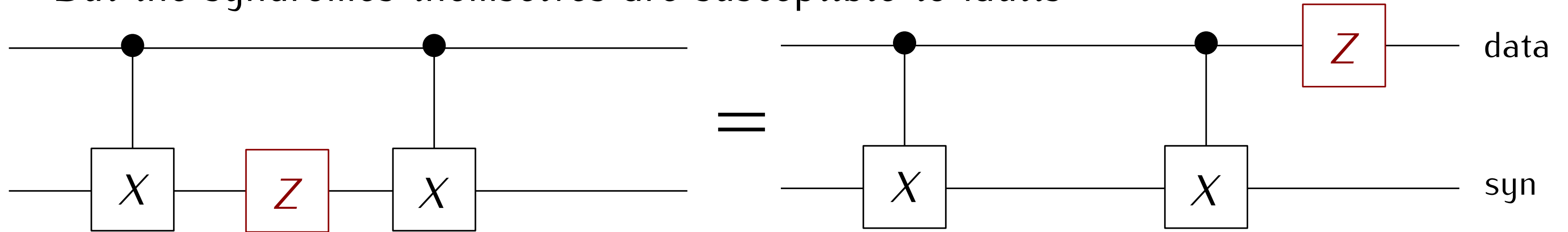
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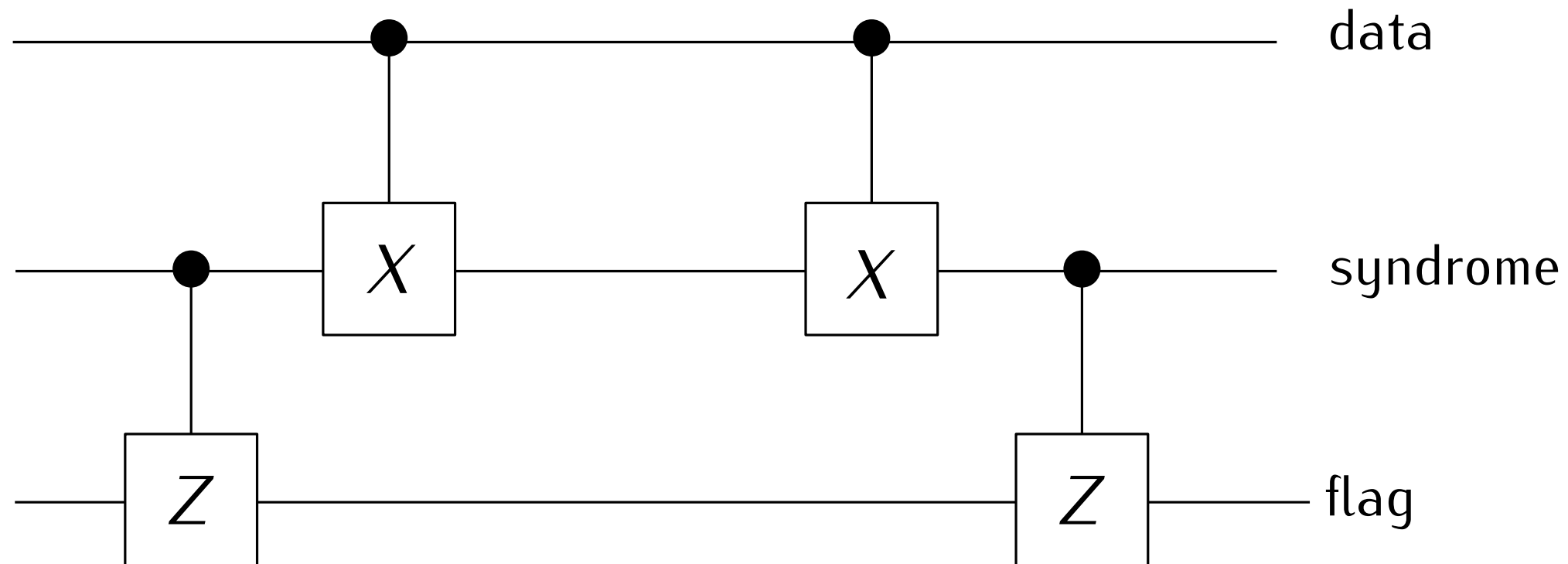


Flags

- But the syndromes themselves are susceptible to faults

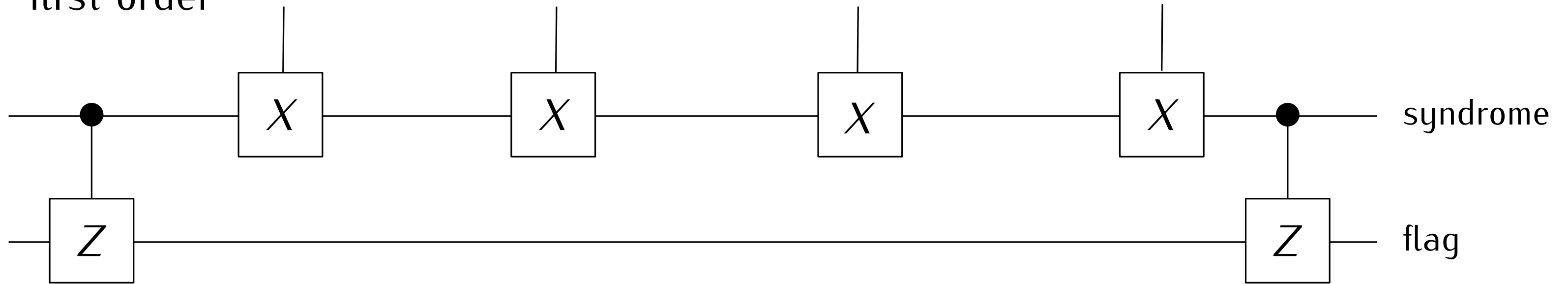


- **Flags:** syndromes for syndromes

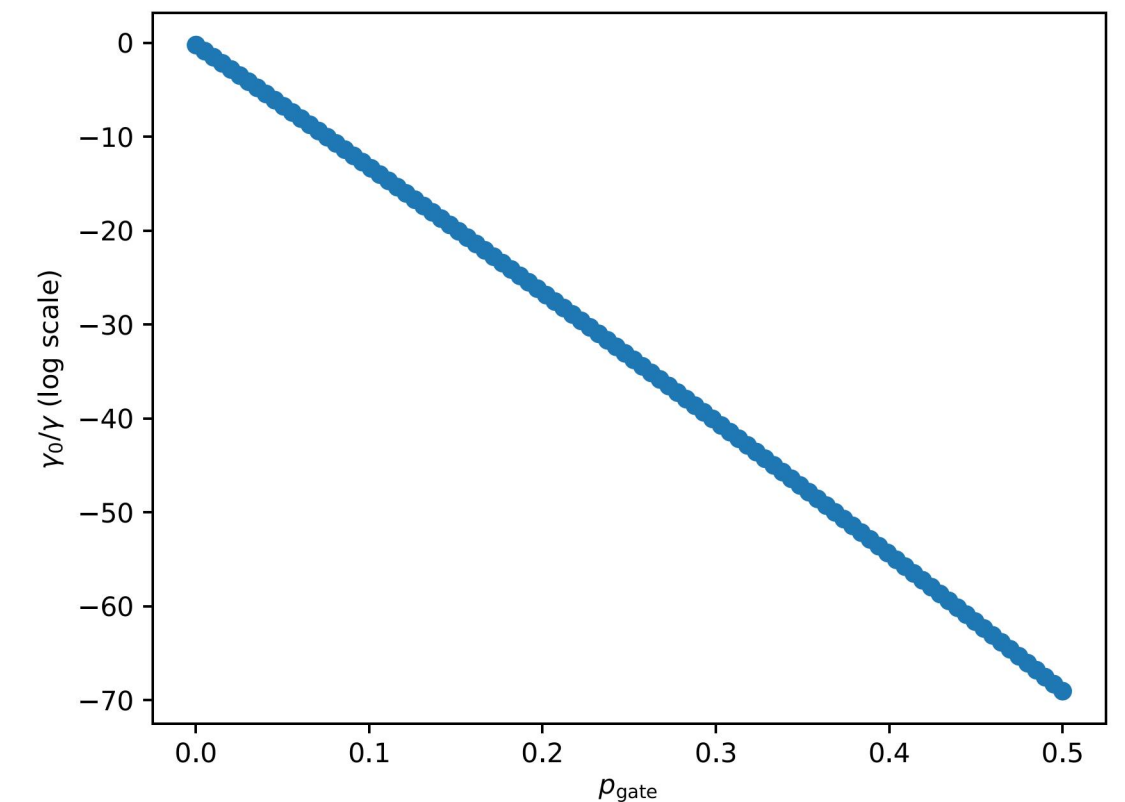


What we found

- [HE, Martiel, Rai, Zenia, to be submitted] Theorem: trivial flags are optimal to first order



- [HE, Martiel, Rai, Zenia, to be submitted]
Experiments: the **noiser** the circuit is, the more it benefits from adding flags



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My contributions

Presented today

[HE & Maria, 2026] *Hardness of computation of quantum invariants on 3-manifolds with restricted topology*, European Symposium of Algorithms & Journal of Applied and Computational Topology.

[HE & Maria, 2026] *Compressed data structures for Heegaard splittings*, International Symposium on Computational Geometry.

[HE & Maria, 2026] *Removing online exponential net search from Solovay-Kitaev*, arXiv & HAL (submitted to Symposium on Discrete Algorithms & Quantum Information Processing).

Others

[HE, 2026] *Theory of Trotter error with variable-weight schedules*, HAL.

[Lu, Hang, HE, Abbassi, 2025] *Variational Quantum Brushes*, arXiv & HAL.

[HE & Tinarrage, 2025] *LieDetect: Detection of representation orbits of compact Lie groups from point clouds*, Foundations of Computational Mathematics.

[Tinarrage, HE, Resck, Gomes, Pociano, Poco 2025] *Empirical analysis of Biding Precedent efficiency in the Brazilian Supreme Court via Similar Case Retrieval*, Artificial Intelligence and Law.

Perspectives

- Invert Heegaard diagrams?
- $\#P$ -hardness for homology spheres?
- Fully efficient SK
- Use symmetries for flags?
- Implement Alagic-Lo!

Merci

(don't clap, it's bad luck)